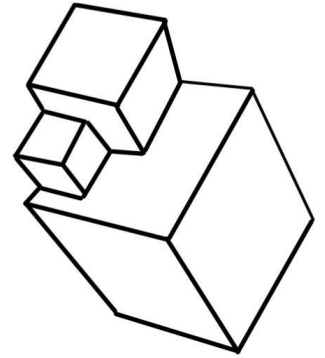


1. **36** From the given information: $3x + 5x + 7x + 9x = 4 \cdot 24 \rightarrow 24x = 96 \rightarrow x = 4$. The largest of the four integers is 36.
2. **17** Re-write the given expression as: $3\sqrt{4}\sqrt{3} + \sqrt{9}\sqrt{3} + \sqrt{25}\sqrt{3} = 6\sqrt{3} + 3\sqrt{3} + 5\sqrt{3} = 14\sqrt{3}$. Then the required sum is 17.
3. **9** $(5x^2 - 3x - 11) - (-3x^2 - 4x + 7) = (5x^2 - 3x - 11) + (3x^2 + 4x - 7) = 8x^2 + x - 18 \rightarrow |a + b + c| = |8 + 1 - 18| = |-9| = 9$. If a student subtracted in the reverse order because it is allowed from the wording of the question, the student would calculate the opposite of our difference. However, the answer would be the same.
4. **4** Re-write the given equation: $3x^2 = 12x - 5 \rightarrow 3x^2 - 12x + 5 = 0$; The sum of the roots $= -\frac{b}{a} = -\frac{-12}{3} = 4$.
5. **128** Use the fact that the measures of the base angles of the isosceles triangle are equal and the fact that the sum of the measures of the interior angles of the triangle is 180 degrees to yield: $2x^2 + 2x + 10 + 2x + 10 = 180 \rightarrow 2x^2 + 4x + 20 = 180 \rightarrow x^2 + 2x + 10 = 90 \rightarrow x^2 + 2x - 80 = 0 \rightarrow (x + 10)(x - 8) = 0 \rightarrow x = -10$ or $x = 8$. $m < C = 2 \cdot 8^2 = 128$.
6. **12** In slope-intercept form, the equations are $y = \frac{2}{3}x + \frac{1}{3}$ and $y = -\frac{a}{8}x + \frac{1}{2}$. Since the lines are perpendicular to each other, their slopes are opposite reciprocals whose product is -1 . Then, $\frac{2}{3}(\frac{-a}{8}) = -1 \rightarrow -2a = -24 \rightarrow a = 12$.
7. **10** $|r^s - s^t + t^r| = |(-2)^3 - 3^1 + 1^{-2}| = |-8 - 3 + 1| = |-10| = 10$.
8. **45** Let x = the number of correct answers and $80 - x$ = the number of incorrect answers. $38 = x - 0.2(80 - x) \rightarrow 380 = 10x - 2(80 - x) \rightarrow 380 = 10x - 160 + 2x \rightarrow 540 = 12x \rightarrow x = 45$.
9. **24** The area of semicircle A is $(4^2\pi)/2 = 8\pi$. The area of semicircle B is $(6^2\pi)/2 = 18\pi$. The diameter of circle O is $4 + 4 + 6 + 6 = 20$. The area of semicircle O is $\frac{(10^2\pi)}{2} = 50\pi$. The area of the shaded region is the area of semicircle O - (area of semicircle A + area of semicircle B) = $50\pi - (8\pi + 18\pi) = 24\pi \rightarrow a = 24$.
10. **7** The units digit of the powers of 23 are: 1, 3, 9, 7, 1, 3, 9, 7... repeating in groups of 4. Then, $97/4 = 24$ r1 and the units digit of 23^{97} is the same as 23^1 or 3. The units digit of the powers of 17 are: 1, 7, 9, 3, 1, 7, 9, 3...repeating in groups of 4. Then, $98/4 = 24$ r2 and the units digit of 17^{98} is the same as 17^2 or 9. The units digit of the powers of 29 are: 1, 9, 1, 9... repeating in groups of 2. Then, $92/2 = 46$ r0 and the units digit of 29^{92} is the same as 29^0 or 1. The product of the units digits of the 3 factors is $3 \cdot 9 \cdot 1 = 27$ and the units digit of the product is 7.
11. **20** From the given information: $bh = 18$; $b^2 + h^2 = 64$. Then, $(b + h)^2 = b^2 + 2bh + h^2 = 64 + 2 \cdot 18 = 100 \rightarrow b + h = 10$. Double that sum and the perimeter of the rectangle is 20.
12. **32** The prime factors of 15015 are 3, 5, 7, 11, and 13. Since each prime factor can appear either once or not at all in each divisor, there are $2^5 = 32$ possible positive integer divisors.

Alternatively, the solution can be viewed as the sum of all possible product combinations of the 5 prime factors: $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 1 + 5 + 10 + 10 + 5 + 1 = 32$.

13. **102** Let a = Akash's age, b = Ben's age and c = Chia's age. The given information yields: $a - 16 = b + c$ and $a^2 = (b + c)^2 + 1632 \rightarrow a^2 = (a - 16)^2 + 1632 \rightarrow a^2 = a^2 - 32a + 256 + 1632 \rightarrow 32a = 1888 \rightarrow a = 59 \rightarrow b + c = 59 - 16 = 43$. The sum of the three ages is $59 + 43 = 102$.
14. **90** The region is a square whose vertices are $(\pm 3\sqrt{5}, 0)$, and $(0, \pm 3\sqrt{5})$ and the length of whose diagonals are each $6\sqrt{5}$. A square is a rhombus, and its area is one-half the product of the lengths of its diagonals. Therefore, the area is $\frac{1}{2}(6\sqrt{5})(6\sqrt{5}) = 90$.
15. **604** Use the fact that the surface area of a cube is the product of 6 and the square of the edge-length. The surface areas of the three cubes in square centimeters are 54, 150, and 486. Note that the minimum surface area is achieved when the 3-centimeter cube and the 5-centimeter cube are glued to each other and to the SAME face of the 9-centimeter cube. When the 3 centimeter and 5-centimeter cubes are glued together, 9 square centimeters are lost from each cube. When the 3 centimeter and 9-centimeter cubes are glued together, another 9 square centimeters are lost from each cube. Finally, when the 5 centimeter and 9-centimeter cubes are glued together, another 25 square centimeters are lost from each cube. Thus, the area of the new figure in square centimeters is $(54 + 150 + 486) - (9 + 9 + 9 + 9 + 25 + 25) = 604$.



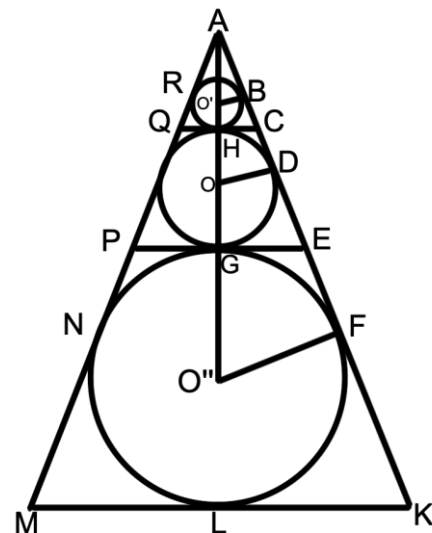
Grade Level 10 – NMT 2024

Solutions

1. **24** There are four possibilities for the first letter, three for the second, two for the third, and one for the fourth. The counting principle requires multiplication: $4 \cdot 3 \cdot 2 \cdot 1 = 24$.
2. **25** The area of the triangle is maximized if point C is as far away from the diameter as possible because that will make for the largest possible altitude to the diameter, which is the base of the triangle. That creates an isosceles right triangle with base of 10, altitude 5, and area of 25.
3. **65** From the given, $4 = 64r^4 \rightarrow r^4 = \frac{1}{16} \rightarrow a_{15} = a_7 r^8 = 4 \cdot \left(\frac{1}{16}\right)^2 = \frac{1}{64}$. The required sum is 65.
4. **3** The vertex of the parabola: $f(x) = x^2 - 4x + 7$ is the point with coordinates $(2, 3)$. Since the parabola has a minimum point, the minimum value of $f(x)$ is 3.
5. **32** From the given, $f(2) = f(1 + 1) = f(1)f(1) = 2 \rightarrow f(1) = \sqrt{2}$. Then, $f(5) = f(1 + 1 + 1 + 1 + 1) = [f(1)]^5 = (\sqrt{2})^5 = k \rightarrow k^2 = ((\sqrt{2})^5)^2 = 32$.
6. **675** From points A and B , drop perpendiculars to the x -axis to create two $30^\circ - 60^\circ$ right triangles. From that diagram, $m \angle AOB = 150^\circ \rightarrow \tan \theta = -\frac{\sqrt{3}}{3} = k \rightarrow 2025k^2 = 675$.

7. **60** Let Joe's average walking speed be r mph. Then, $\frac{1}{r} + \frac{8}{r+54} = \frac{18}{60} \rightarrow \frac{9r+54}{r(r+54)} = \frac{3}{10} \rightarrow 90r + 540 = 3r^2 + 162r \rightarrow 3r^2 + 72r - 540 = 0 \rightarrow r^2 + 24r - 180 = 0 \rightarrow (r+30)(r-6) = 0 \rightarrow r = 6$. The average speed of the bus is 60 mph.
8. **157** The first digit must be 1 or 3 because there are five odd digits. It cannot start with 3 because 3579 is divisible by 3. Therefore, it starts with 1. The possibilities are 1359, 1379, and 1579. 1359 is divisible by 3 and 1379 is divisible by 7. Therefore, the passcode is 1579. Thus, the three-digit number representing the first three digits of the passcode is 157.
9. **16** Rewrite the given: $(1-i)^9 = (1-i)[(1-i)^2]^4 = (1-i)(-2i)^4 = (1-i)16 = 16-16i$. The real part is 16.
10. **103** Note that the successive differences increase by 6. Thus, $f(5) = 47 + 25 = 72$ and $f(6) = 72 + 31 = 103$. Alternatively, use the first three function values for $f(x) = ax^2 + bx + c \rightarrow \{8 = a + b + c, 15 = 4a + 2b + c, 28 = 9a + 3b + c\} \rightarrow \{7 = 3a + b, 13 = 5a + b\} \rightarrow 2a = 6 \rightarrow a = 3, b = -2, c = 7 \rightarrow f(6) = 3 \cdot 36 - 2 \cdot 6 + 7 = 103$.
11. **76** From the given: $P(x) = Q(x)(x-1)(x-3) + ax + b \rightarrow P(1) = a + b = 4$ and $P(3) = 3a + b = -4 \rightarrow a = -4, b = 8 \rightarrow a + 10b = 76$.
12. **14** Divide the given equation by 2: $\frac{\frac{\sqrt{3}}{2}}{\sin x} + \frac{\frac{1}{2}}{\cos x} = 2 \rightarrow \frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x = 2 \sin x \cos x \rightarrow \sin\left(\frac{\pi}{3} + x\right) = \sin 2x \rightarrow \frac{\pi}{3} + x = 2x$ or $\frac{\pi}{3} + x = \pi - 2x \rightarrow x = \frac{\pi}{3}$ or $\frac{2\pi}{9}$. The sum of the two solutions is $\frac{5\pi}{9}$ and the required sum is 14.
13. **16** The given pair of equations can be combined into: $|x-4| + x-4 = 2$. Then, if $x \geq 4, 2x-8 = 2 \rightarrow x = 5 \rightarrow 5 = |y-3| + 4 \rightarrow |y-3| = 1 \rightarrow y = 4$ or 2 . If $x < 4, 0 = 2$. Of course, this is impossible. The required solution is the ordered pairs: (5,2) and (5,4). The required sum is 16.
14. **13** From the given, $(CB)^2 = 4(AC)^2 \rightarrow (x-4)^2 + y^2 = 4[(x-1)^2 + y^2] \rightarrow x^2 - 8x + 16 + y^2 = 4x^2 - 8x + 4 + 4y^2 \rightarrow 3x^2 + 3y^2 = 12 \rightarrow x^2 + y^2 = 4$. This equation describes a circle with radius of 2 and an area of $4\pi \approx 13$.

15. **97** Since $EG = ED = 5$, and $AE = 13$, then $AD = 13 - 5 = 8$. From the diagram, $\triangle AGE \sim \triangle ADO \rightarrow \frac{AG}{GE} = \frac{AD}{OD} \rightarrow \frac{12}{5} = \frac{8}{OD} \rightarrow OD = \frac{10}{3}$. $AH = 12 - \frac{20}{3} = \frac{16}{3}$. Since $\triangle ACH \sim \triangle AEG$, $AH:AG = \frac{16}{3}:12 = 4:9$. Draw one additional line tangent to the lower circle at point L to create three similar triangles. Using the letters indicated in the figure, $\triangle AQC \sim \triangle APE \sim \triangle AMK$. Let point K on the extension of $\overline{AE}$ be a vertex of the largest of the three triangles. Consider the dilation with center point A that maps point E to point K . The middle and largest triangles along with their circles are the images of the smaller and middle triangles and their circles. Therefore $\frac{O'B}{OD} = \frac{OD}{O''F} = \frac{4}{9} \rightarrow \frac{O'B}{OD} \cdot \frac{OD}{O''F} = \frac{16}{81}$. The required sum is 97.



1. **175** There are $4x$ students who own both a scientific calculator and a graphing calculator. There are $5x$ students who own only a graphing calculator. Then, $9x = 225 \rightarrow x = 25 \rightarrow 4x = 100$ and $5x = 125$. There are $300 - 225 = 75$ students who own only a scientific calculator and 100 students that own both. Therefore $100 + 75 = 175$ students own a scientific calculator.
2. **18** The given expression $\frac{4^x}{8^{12}} = \frac{2^{2x}}{2^{36}} = 2^{2x-36}$. In order for 2^{2x-36} to be a square of an integer, the exponent must be an even integer. Therefore, $2x - 36 = 0 \rightarrow x = 18$.
3. **147** By the midsegment theorem, the length of one side of the equilateral triangle is 14. The perimeter of the triangle is 42 and the semi-perimeter is 21. By Heron's formula, $A = \sqrt{21(21-14)(21-14)(21-14)} = \sqrt{3(7)(7)(7)(7)} = \sqrt{3 \cdot 7^4} = 7^2\sqrt{3} = 49\sqrt{3}$. So, the required product is $49 \cdot 3 = 147$.
4. **738** The sum of an arithmetic sequence is $S_n = \frac{n(a_1+a_n)}{2}$ and $\frac{a_1+a_n}{2}$ is the median of the sequence. Therefore, $S_{123} = 123(6) = 738$.
5. **4** Since $2x - 1$ is not a factor of the numerator, an equation of the vertical asymptote is $x = \frac{1}{2}$. An equation of the oblique asymptote can be found by division: $(3x^2 + 5x - 1) \div (2x - 1) = \frac{3}{2}x + \frac{13}{4} + \frac{9/4}{2x-1}$. Thus, an equation of the oblique asymptote is $y = \frac{3}{2}x + \frac{13}{4}$, because $\frac{9/4}{2x-1}$ approaches zero as $x \rightarrow \pm\infty$. The asymptotes intersect at $y = \frac{3}{2}\left(\frac{1}{2}\right) + \frac{13}{4} = \frac{3}{4} + \frac{13}{4} = \frac{16}{4} = 4$.
6. **10** Define the smaller side of the rectangle to be $2x$, the radius of the half-circle to be x , and the radius of the quarter-circle to be $2x$. Draw a line segment from the center of the half-circle through the point of tangency to the center of the quarter-circle (the lower right-hand vertex of the rectangle) to create a right triangle. We can be assured that the three points are collinear because each radius is perpendicular to the common tangent line (not drawn). Use the Pythagorean Theorem: $x^2 + (2\sqrt{2})^2 = (3x)^2 \rightarrow x^2 + 8 = 9x^2 \rightarrow 8 = 8x^2 \rightarrow x = 1$. Therefore, the smaller side of the rectangle is 2 and the perimeter is $4 + 4\sqrt{2}$. So, the required sum is $4 + 4 + 2 = 10$.
7. **13** Add the first two equations to yield: $2024x + 2024y = 10,120 \rightarrow x + y = 5 \rightarrow x^2 + 2xy + y^2 = 25 \rightarrow x^2 + 12 + y^2 = 25 \rightarrow x^2 + y^2 = 13$. Also, subtracting the first two equations yields the same result.
8. **63** Interchange x and y and solve for the "new" y to get the inverse function. $f^{-1}(x) = \frac{15x-63}{x-1}$. Set the function equal to its inverse: $\frac{15x-63}{x-1} = \frac{x-63}{x-15} \rightarrow 15x^2 - 289x + 945 = x^2 - 64x + 63 \rightarrow 14x^2 - 225x + 882 = 0 \rightarrow$ the product of the roots is $\frac{882}{14} = 63$. Alternatively, since $f(x) = f^{-1}(x)$, the output of one function is the input of the other: $\frac{x-63}{x-15} = x \rightarrow x - 63 = x^2 - 15x \rightarrow x^2 - 16x + 63 = 0$. Therefore, the product of the roots is 63.

9. **15** Factor the given expression to yield: $\sqrt{300} \left(\cos^2 \left(\frac{\pi}{12} \right) - \sin^2 \left(\frac{\pi}{12} \right) \right) \left(\cos^2 \left(\frac{\pi}{12} \right) + \sin^2 \left(\frac{\pi}{12} \right) \right)$. Use trigonometric identities to simplify this to: $\sqrt{300} \cdot \cos \left(2 \cdot \frac{\pi}{12} \right) \cdot 1 = \sqrt{300} \cdot \frac{\sqrt{3}}{2} \cdot 1 = \frac{\sqrt{900}}{2} = \frac{30}{2} = 15$.
10. **5** Use the properties of logarithms to yield: $\log(10x + 50) - \log(x - 4) = \log \left(\frac{10x+50}{x-4} \right) \geq 2 \rightarrow \frac{10x+50}{x-4} \geq 100$. Since the domain of the original inequality is $x > 4$, the denominator $x - 4$ will always be positive. This allows for multiplication by $x - 4$ without changing the order of the inequality. Thus, $10x + 50 \geq 100x - 400 \rightarrow 450 \geq 90x \rightarrow 5 \geq x$. Combine this solution with the domain to give 5 as the only possible integer.
11. **3** Re-write the given equation as: $bi = a^2 - 2abi - b^2$. Set the real parts equal to each other and the imaginary parts equal to each other: $a^2 - b^2 = 0$ and $b = -2ab$. Re-write the second equation: $b + 2ab = 0 \rightarrow b(1 + 2a) = 0 \rightarrow b = 0$ or $a = -0.5$. If $b = 0$, then, from the first equation: $a = 0$. If $a = -0.5$, then, from the first equation: $0.25 - b^2 = 0 \rightarrow b = \pm 0.5$. Therefore b can be equal to 0, 0.5 or -0.5 . Thus, there are 3 real values of b that satisfy the given equation.
12. **90** Ms. Lovelace grades at a rate of $100/3$ tests per hour and Mr. Pascal grades at a rate of 40 tests per hour. The following equation describes the problem situation, where t denotes the time (in hours) that Mr. Pascal spends grading the contests: $40t + \frac{100}{3}(t + 5) = 6400 \rightarrow 220t + 500 = 19200 \rightarrow 220t = 18700 \rightarrow t = 85 \text{ \& } t + 5 = 90$.
13. **18** The equation will be satisfied if $x^2 = 2x$. However, the solutions of 0 and 2 do not satisfy the additional condition, namely $x > 2$. Without loss of generality, the equation will also be satisfied if x^2 is in quadrant IV and $2x$ is in quadrant I. Therefore, let $x^2 = 360 - 2x \rightarrow x^2 + 2x - 360 = 0 \rightarrow (x + 20)(x - 18) = 0 \rightarrow x = 18$.
14. **4** In right $\triangle ABC$, the Pythagorean Theorem yields: $(2\sqrt{2x})^2 + 4^2 = AC^2$. In $\triangle APC$, the Law of Cosines yields: $x^2 + (2x)^2 - 2 \cdot x \cdot 2x \cos 60^\circ = AC^2 \rightarrow 8x + 16 = 5x^2 - 4x^2 \cdot \frac{1}{2} \rightarrow 8x + 16 = 3x^2 \rightarrow 0 = 3x^2 - 8x - 16 \rightarrow 0 = (3x + 4)(x - 4) \rightarrow x = 4$.
15. **253** Emmy could have randomly picked 5 of her 10 homework problems in $\binom{10}{5} = \frac{10!}{5!5!} = 252$ ways. There is exactly one way that her teacher can pick the exact same 5 problems to grade. So, the probability that her teacher grades the exact problems that Emmy did for homework is $\frac{1}{252}$ and the required sum is 253.

Grade Level 12 – NMT 2024

Solutions

1. **46** $2024 = 2025 - 1 = 45^2 - 1^2 = (45 - 1)(45 + 1) = 44 \cdot 46$, so $2024 \div 44 = 46$.
2. **55** Add 1 to each side to complete the square: $x^2 - 20x + 100 = 2025 \rightarrow (x - 10)^2 = 45^2 \rightarrow x - 10 = \pm 45 \rightarrow x = 10 \pm 45$. The greater root is 55.

3. **600** Note that Art costs $\frac{22 \text{ dollars}}{1 \text{ hour}} \cdot \frac{1 \text{ hour}}{5 \text{ cakes}} = \frac{22}{5} = \4.40 per cake. Computing similarly, Helen costs $\frac{30}{7} = \$4.28$ per cake and Pete costs $\frac{35}{8} = \$4.38$ per cake. Helen is the least expensive option and Curt should hire only her, at a cost of $140 \cdot \frac{30}{7} = \600 .
4. **42** If $x = 0$, the y -intercept is $(0,12)$. Note that by factoring, $y = \frac{1}{12}(x-9)(x-16)$, and if $y = 0$, then the x -intercepts are $(9,0)$ and $(16,0)$. Using either the distance formula or Pythagorean theorem on these three points results in side lengths of 7, 15, and 20, for a total perimeter of 42.
5. **675** Inscribe the octagon in a circle, noting that each side of the octagon subtends a 45° arc. Then, using the formula for the angle between chords in a circle, $m\angle FIH = \frac{1}{2}(m\widehat{FGH} + m\widehat{AB}) = \frac{1}{2}(90^\circ + 45^\circ) = 67.5^\circ$. The required product is 675.
6. **4** There are $\binom{9}{2} = \frac{9 \cdot 8}{2} = 36$ possible pairs of squares that can be chosen. There are 6 pairs of horizontally adjacent squares (all but the last column of squares has adjacent squares to the right). Similarly, there are 6 pairs of vertically adjacent squares. Thus, the probability is $\frac{12}{36} = \frac{1}{3}$, and the required sum is 4.
7. **136** The horizontal asymptote can be obtained by dividing the dominant terms in the numerator and denominator: $y = \frac{6x^2}{x^2} = 6$. Note that $f(x) = \frac{6x(x+7)}{(x-13)(x+7)}$, which has a hole at $x = -7$ and a vertical asymptote at $x = 13$, so the asymptotes intersect at $(13, 6)$, and the required sum is 136.
8. **322** Without loss of generality, let the square have a side length of 2, so the area of each large circle is π . Since the diagonal of the square is $2\sqrt{2}$, the radius of the small circle is $\frac{1}{2}(2\sqrt{2} - 2) = \sqrt{2} - 1$. So, the area ratio is $\frac{\pi}{\pi(\sqrt{2}-1)^2} = \frac{1}{3-2\sqrt{2}} = 3 + 2\sqrt{2}$, giving us an answer of 322.
9. **617** Since $\sin^2 x + \cos^2 x = 1$, $\cos x = \pm\sqrt{1 - \sin^2 x}$. Since $\frac{\pi}{6}$ is in quadrant I, $\sqrt{1 - \sin^2 x} = \cos x$. So, $f(x) = 1234 \sin x \cos x = 617 \sin 2x$. Thus, $f'(x) = 1234 \cos 2x$ and $f'\left(\frac{\pi}{6}\right) = 1234 \cdot \frac{1}{2} = 617$.
10. **85** Let K be the area of the triangle, $BC = a$, and $AC = b$. Then, $K = \frac{1}{2}ab \sin 75^\circ$ and $\frac{dK}{dt} = \frac{1}{2} \sin 75^\circ \left(a \frac{db}{dt} + b \frac{da}{dt}\right)$. Since K must be constant, $\frac{dK}{dt} = 0$, and so $a \frac{db}{dt} + b \frac{da}{dt} = 0 \rightarrow \frac{da}{dt} = -\frac{a}{b} \frac{db}{dt} = -\frac{20}{24}(-102) = 85$.
11. **179** To find intersections, equate the functions to get $x^2 + 2024 = mx \rightarrow x^2 - mx + 2024 = 0$. If the graphs do not intersect, that equation has no real roots, so $b^2 - 4ac = m^2 - 4(2024) < 0 \rightarrow m^2 < 4 \cdot 2024 \rightarrow -2\sqrt{2024} < m < 2\sqrt{2024}$. Since $2025 = 45^2$, and $\sqrt{2024}$ is very close to 45, $-89 \leq m \leq 89$, so m has $2(89) + 1 = 179$ possible values.

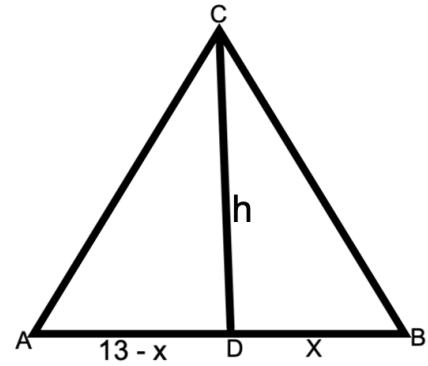
12. **35** Method 1: $|z_1 z_2 - z_1|$ is the distance between $z_1 z_2$ and z_1 in the complex plane. Call that distance d . Note that $|z_1| = \sqrt{24^2 + 7^2} = 25$. Since $z_2 = 0.6\text{cis}120^\circ$, $|z_1 z_2| = 0.6(25) = 15$. Consider the triangle formed by 0, z_1 , and $z_1 z_2$ in the complex plane. Two sides of the triangle have lengths of 25 and 15, as we found. The angle between those sides must be 120° , and the length of the third side, d , can be found by the Law of Cosines: $d^2 = 15^2 + 25^2 - 2(15)(25)\cos 120^\circ \rightarrow d = 35$.
- Method 2: $|z_1 z_2 - z_1| = |z_1||z_2 - 1| = |24 + 7i||-1.3 + 0.3\sqrt{3}i| = \sqrt{24^2 + 7^2} \cdot \sqrt{(-1.3)^2 + (0.3\sqrt{3})^2} = \sqrt{625} \cdot \sqrt{1.69 + 0.27} = 25\sqrt{1.96} = 25(1.4) = 35$.
13. **40** The two graphs of any function and its inverse are reflections of each other over the line $y = x$. Since these two graphs intersect, the intersection point is on the line whose equation is $y = x$. If they intersect only one time, then each curve touches the line $y = x$ only once. Therefore, each curve must be tangent to the line $y = x$. So, $y = ae^x$ must be tangent to the line at a point on the line which can be called (c, c) . Since this point is on the curve $y = ae^x$, $c = ae^c$. Because the tangent line is $y = x$, $f'(c) = 1$ and $ae^c = 1, \rightarrow c = 1$. Thus, $ae = 1 \rightarrow a = \frac{1}{e}$. So, $108a \approx 108\left(\frac{1}{2.7}\right) = \frac{1080}{27} = 40$. (This is accurate to the nearest integer.)
14. **600** Look at the beginning of the sequence: $0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}, \frac{21}{32}, \frac{43}{64}, \frac{85}{128}, \dots$ You might notice that the numerators are approximately $2/3$ of the denominators, which implies that the limit of the sequence might be $2/3$. To confirm this, note that if the numerator is n and the denominator is d , that $3n \pm 1 = 2d$ for every term of the sequence, so $n = \frac{2d \pm 1}{3} = \frac{2}{3}d \pm \frac{1}{3}$ and $a_n = \frac{n}{d} = \frac{\frac{2}{3}d \pm \frac{1}{3}}{d} = \frac{2}{3} \pm \frac{1}{3d}$. As n approaches infinity, d approaches infinity as well (look at the list above), so $\lim_{n \rightarrow \infty} a_n = \lim_{d \rightarrow \infty} \left(\frac{2}{3} \pm \frac{1}{3d}\right) = \frac{2}{3}$ and $900 \cdot \frac{2}{3} = 600$.
15. **506** For a large number of trials, the expected value of $R(k)$ is $\frac{1+k}{2}$, so, as $n \rightarrow \infty$, $\sum_{k=1}^n R(k)$ will approach $\sum_{k=1}^n \frac{1+k}{2} = \frac{1}{2}(\sum_{k=1}^n 1 + \sum_{k=1}^n k) = \frac{1}{2}\left(n + \frac{n(n+1)}{2}\right) = \frac{1}{2}\left(\frac{n^2+3n}{2}\right) = \frac{n^2+3n}{4}$. So, $2024 \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n R(k)}{n^2} = 2024\left(\frac{1}{4}\right) = 506$.

Mathletics - NMT 2024

Solutions

- 36** Factor $2024 = 11 \cdot 23 \cdot 2^3 \rightarrow r = 2$ and $p + q + r = 36$.
- 30** Solve the system of equations $4.5k + 5.25v = 261$ and $k = v + 6 \rightarrow 18(v + 6) + 21v = 1044 \rightarrow 39v + 108 = 1044 \rightarrow 39v = 936 \rightarrow v = 24$ and $k = 30$.
- 64** Re-write the second equation as $y = (x - 20)^2 + 100$. Notice that this parabola has a vertex at $(20, 100)$ and is a minimum point. The first parabola has a vertex at $(20, 36)$ and has a maximum point. Thus, the minimum distance between the parabolas is the distance between the two vertices and $100 - 36 = 64$.

4. **181** Method 1: Compute the area of the triangle using Heron's formula (where s = semi-perimeter): $A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21(21-13)(21-14)(21-15)} = 84$. The longest altitude of the triangle is the altitude drawn to the side of length 13. Then, $84 = \frac{1}{2}bh \rightarrow 84 = \frac{1}{2}(13)h \rightarrow h = \frac{168}{13}$. The required sum is 181.



Method 2: In the accompanying diagram of $\triangle ABC$ draw altitude $\overline{CD}$ and use the Pythagorean theorem to write two equations: $x^2 + h^2 = 14^2$ and $(13-x)^2 + h^2 = 15^2 \rightarrow (13-x)^2 - x^2 = 29 \rightarrow x = \frac{70}{13} \rightarrow h = \frac{168}{13}$. The required sum is 181.

5. **221** The probability that Sam wins 5 or more rounds is the sum of the probabilities that he wins exactly 5 or exactly 6 or exactly 7 or exactly 8 rounds. Let $P(n)$ represent probability that Sam wins n rounds out of 8. The formula for Bernoulli trials to calculate each probability is :

$$P(n) = \binom{n}{r} p^r (1-p)^{n-r}.$$

$$P(5) = \binom{8}{5} \left(\frac{3}{7}\right)^5 \left(\frac{4}{7}\right)^3$$

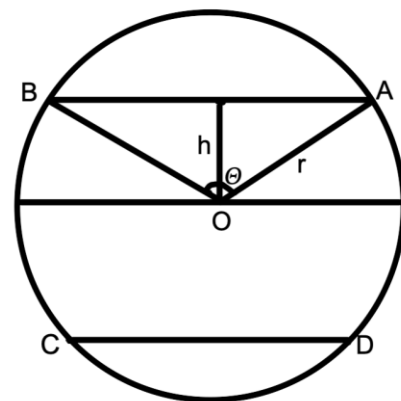
$$P(6) = \binom{8}{6} \left(\frac{3}{7}\right)^6 \left(\frac{4}{7}\right)^2$$

$$P(7) = \binom{8}{7} \left(\frac{3}{7}\right)^7 \left(\frac{4}{7}\right)^1$$

$$P(8) = \left(\frac{3}{7}\right)^8$$

The product of 1000 and the sum of these probabilities rounded to the nearest integer is 221.

6. **265** The area of the minor segment associated with chord $\overline{AB}$ is one-third the area of the circle. The area of the minor segment associated with chord $\overline{AB}$ is equal to the area of sector AOB minus the area of $\triangle AOB \rightarrow \frac{1}{3}\pi r^2 = \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta \rightarrow \frac{1}{3}\pi = \frac{1}{2}\theta - \frac{1}{2}\sin \theta$. Note that θ measured in radians.



is

This equation cannot be solved algebraically, but it can be graphed and the result is an approximate solution: $\theta \approx 2.6053$ radians. Since $r = 500$, $h = 500\cos\left(\frac{\theta}{2}\right) \approx 132.47$. By the symmetry of the problem, the distance between these chords is $2h \approx 265$, rounded to the nearest integer.

7. **480** By setting $x = 0$ and then $y = 0$, it can be determined that the region is a rhombus whose vertices are $(\pm 24, 0)$ and $(0, \pm 10)$. The area of the rhombus is $A = \frac{1}{2}d_1d_2 = \frac{1}{2}(48)(20) = 480$.

8. **15** The sum of the first n terms of an arithmetic sequence is given by $S_n = \frac{n}{2}(a_1 + a_n)$.

$$S_n = 1 + 5 + 9 + \cdots + (4n - 3) = \frac{n(4n - 2)}{2} = 2n^2 - n$$

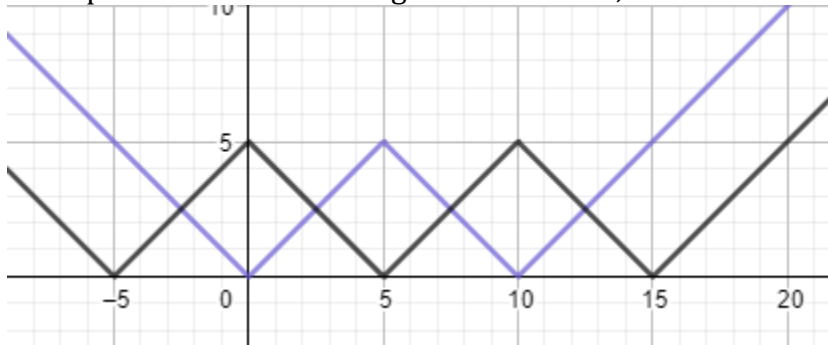
$$f(n) = \sum_{k=1}^n S_k = \sum_{k=1}^n (2k^2 - k) = 2 \cdot \sum_{k=1}^n k^2 - \sum_{k=1}^n k = \frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

We can use a calculator to create a table of values or a graph to determine the value of n .

Alternatively, we can calculate the first few values of $f(n) = \sum_{k=1}^n k(2k - 1)$ by hand:

$f(1) = 1$; $f(2) = 1 + 2 \cdot 3 = 7$; $f(3) = 7 + 3 \cdot 5 = 22$; $f(4) = 22 + 4 \cdot 7 = 50$; and continue with a calculator until $f(15) = 2360 > 2024 \rightarrow t = 15$.

9. **46** Graph the first few functions in the sequence and notice that these graphs create a saw-tooth wave pattern around the origin. For instance, we can see 4 solutions to $f_2(x) = f_3(x)$:



A graph shows that $f_3(x) = f_4(x)$ has 6 solutions and continue in this pattern to note that $f_{23}(x) = f_{24}(x)$ has 46 solutions.

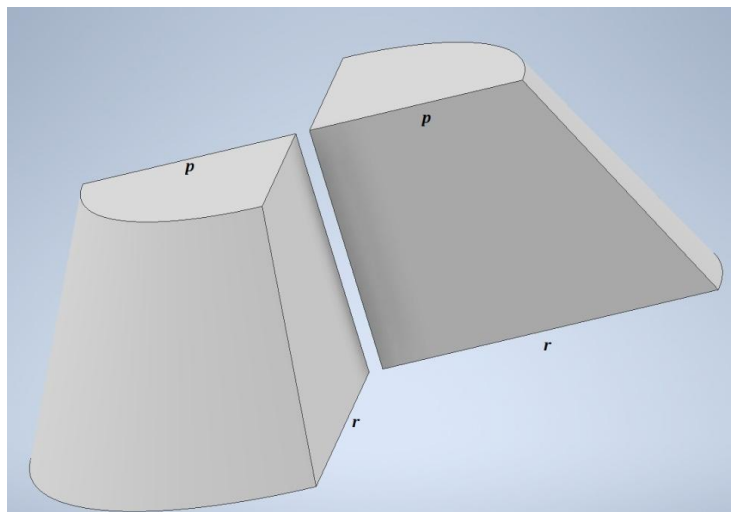
10. **677** The accompanying diagram shows only two of the four pieces formed by the two perpendicular planes. These 4 pieces each have three interior faces: a quarter circle and two congruent trapezoids.

The volume of the original cone is:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (2r) = \frac{2}{3}\pi r^3 = 500 \rightarrow r \approx 6.20350 \text{ and } h = 2r \approx 12.40701.$$

The smaller upper cone is similar to the original cone. So, the ratio of its height to the radius

of its base is also 2:1. Let p be the radius of the smaller cone. Its volume is: $\frac{2}{3}\pi p^3 = 100 \rightarrow p \approx 3.62783$. (Alternatively, $\frac{r}{p} = \sqrt[3]{5}$ is another approach to finding p .) The total surface area of the five objects equals the surface area of the original cone plus twice the surface area of the base of the smaller cone plus the surface area of the 8 trapezoidal faces (4 shown and 4



omitted). The bases of each of the trapezoids are the radii of the two cone bases. The height of each trapezoid is given by the difference of the heights of the cones: $t = 2r - 2p \approx 5.15135$. Note that each cone's altitude is twice the radius of its cone.
 For the main cone: $SA = \pi r^2 + \pi r \sqrt{r^2 + h^2} \approx 391.2386657$.
 The two discs: $SA = 2\pi p^2 \approx 82.69402$.
 Eight trapezoids: $A = 8 \cdot \frac{1}{2} (2r - 2p)(2r + 2p) \approx 202.57848$.
 The sum of the three surface areas is $676.5111657 \approx 677$

Team Problem Solving - NMT 2024

Solutions

1. **853** The greatest single digit factor of 120 is 8. Let 8 be the digit in the hundred's column. Since $120 = 8 \times 15$, the remaining two digits must be 5 and 3. The greatest 3-digit integer the product of whose digits is 120 is 853.
2. **127** Since $f(0) = 1, f(1) = 2f(0) + 1 = 3$. Similarly, $f(2) = 2f(1) + 1 = 7, f(3) = 2f(2) + 1 = 15, f(4) = 2f(3) + 1 = 31, f(5) = 2f(4) + 1 = 63$, and $f(6) = 2f(5) + 1 = 127$. Test to see if 127 is prime, which it is. The values of $f(n)$ form numbers that are 1 less than powers of 2. So, the remaining 3-digit values of $f(n)$ are 255 and 511. Neither of these are primes. So, the greatest 3-digit prime number that fits the conditions of the problem is 127.
3. **73** Recognize that 6351 is nearly the perfect square 6400. That is, $6351 = 6400 - 49 = 80^2 - 7^2 = (80 + 7)(80 - 7) = 87 \times 73 = 3 \cdot 29 \cdot 73$. The greatest of these 3 prime factors is 73.
4. **3** Let p = the number of pennies, n = the number of nickels, and d = the number of dimes. Then $p + n + d = 105$ and $2(p + 5n + 10d) = \frac{p}{2} + 20n + 20d$. Simplify the second equation to $3p = 20n$. Many ordered pairs satisfy this equation but since all answers must be prime, consider only prime numbers for n . When $n = 2$, p is not an integer. When $n = 3$, $p = 20$ and $d = 82$. This is the only possible solution, since n must be a multiple of 3 and it must be prime.
5. **503** Notice that N cannot equal D, so N must equal D + 1. This implies that I must be 9 so I + D will have a sum of 10 or more and O will equal 0. Since NOT must be a prime number, D cannot be odd since T cannot be even. We want the least prime value for NOT, so try D = $\frac{9}{\text{NOT}}$. The result is 301 which factors into 7×43 . Try D = 4. The result is 503, which is prime.

9	9	9
+ D9D	+ 292	+ 494
NOT	301	503

 [Although 907 is another possible value for NOT, it is not the least value.]
6. **59** Since $a < b < c$ and $2 \leq b \leq 5$, examine all possible cases for b .
 If $b = 2$, then $a = 1$ and $c = 3, 4, 5, 6 \rightarrow 4$ cases.
 If $b = 3$, then $a = 1$ or 2 and $c = 4, 5, 6 \rightarrow 2 \cdot 3 = 6$ cases.
 If $b = 4$, then $a = 1, 2$, or 3 and $c = 5$ or 6 $\rightarrow 3 \cdot 2 = 6$ cases.
 If $b = 5$, then $a = 1, 2, 3$, or 4 and $c = 6 \rightarrow 4$ cases.
 Since there are $6 \times 6 \times 6 = 216$ possible outcomes when 3 dice are rolled, $\frac{p}{q} = \frac{20}{216} = \frac{5}{54}$,
 $p + q = 54 + 5 = 59$.

7. 5 Jill's average rate for running uphill and downhill is her total distance/total time. Since her route from home to work and back is the same, the number of uphill miles and the number of downhill miles are equal. Therefore, her average rate for the round trip is $12/(2 + 1) = 4$ mph. This is the same as her rate on level ground. Thus, her total distance = $4 \times 1.25 = 5$ miles.
8. 59 Let $100h + 10t + 7$ represent the three-digit number with the appended 7 and let $10h + t$ represent the original two digit number. Then, $100h + 10t + 7 = 538 + 10h + t \rightarrow 90h + 9t = 531 \rightarrow 10h + t = 59$. The original two-digit number is 59.
9. 17 Since $acb, a79, b0c, bb1$ form an arithmetic sequence, and since $bb1 > b0c$, the terms are increasing. Let tu be the two-digit common difference between consecutive terms, so $a79 + tu = b0c$. It follows that $9 + u = c + 10 \rightarrow u = c + 1$. It also follows that $c + (c + 1) = 11 \rightarrow c = 5$ & $u = 6$. Since $a79 + t6 = b05$, $7 + t + 1 = 10 \rightarrow t = 2$. Finally, $b05 + 26 = b31 \rightarrow b = 3$ & $a = 2 \rightarrow a + bc = 2 + 3 \cdot 5 = 17$.
10. 17 The last possible primedate before 2/2/2024 might be 11/29/2023. Unfortunately, $2023 = 7 \times 17^2$. The number $2021 = 2025 - 4 = (45^2 - 2^2) = 43 \times 47$. So it is not prime. Notice that 2019 has a factor of 3. So it is not prime either. The last primedate was 11/29/2017 and $2017/(11 + 29) = 50$ with a remainder of 17.
11. 19 Let $PM = PC = x$, so $CM = 2x$ and $NC = 6x$. It follows that $QM = 2x$ and $LQ = 4x$. The area of rectangle $NCML = 2x \cdot 6x = 12x^2 = 1 \rightarrow x^2 = \frac{1}{12}$. The area of $\triangle QPM = x^2 = \frac{1}{12}$ and the area of $\triangle QNL = 4x^2 = \frac{4}{12}$. So the area of quadrilateral $NQPC = 1 - (1/12 + 4/12) = 7/12$. The ratio of $[NQPC]$ to $[NCML] = 7/12$ and $7 + 12 = 19$.
12. 5 Multiply and simplify: $(2x - 3)(x - q) + 9 = 2x^2 - (p + 1)x + 3p \rightarrow 2x^2 - (3 + 2q)x + 3q + 9 = 2x^2 - (p + 1)x + 3p \rightarrow (3 + 2q)x + 3q + 9 = (p + 1)x + 3p$. Consequently, $3 + 2q = p + 1$ and $3q + 9 = 3p$. Solve this system of equations to yield $q = 1$ & $p = 4 \rightarrow p + q = 1 + 4 = 5$.
13. 37 Separate the x and y coordinates to find out the amount each change from point A to point B . The value of x goes from 3 to 5 so it increases by 2 and the value of y goes from -1 to 4 so it increases by 5. Since the ratio of BC to AC is $4/5$, the ratio of AB to BC is $1/4$. This means that the x - and y - changes will be 4 times greater when going from point B to point C or x will increase by $4 \times 2 = 8$ and y will increase by $4 \times 5 = 20$. The coordinates of point C will be $x = 5 + 8 = 13$ and $y = 4 + 20 = 24$. The sum of $x + y = 13 + 24 = 37$.
Alternatively, $\frac{BC}{AC} = \frac{4}{5} \rightarrow \frac{AC}{AB} = 5$. then, $x_C = x_A + 5(x_B - x_A)$ and $y_C = y_A + 5(y_B - y_A) \rightarrow x = 3 + 5 \cdot 2 = 13$ and $y = -1 + 5 \cdot 5 = 24$. The required sum is 37.
14. 17 Let x = length, y = width, and z = height. Then, $xyz = 864$, $xy = 72 \rightarrow z = 12$, and $xz = 96 \rightarrow x = 8$ & $y = 9$. The length of a diagonal of a right rectangular prism is given by the formula: $d = \sqrt{x^2 + y^2 + z^2} \rightarrow d = \sqrt{64 + 81 + 144} = \sqrt{289} = 17$.
15. 23 Apply the given trig formula: $(\cos x)(\cos y) = \frac{\cos(x+y) + \cos(x-y)}{2}$.
 $(\cos 40^\circ)(\cos 80^\circ) = \frac{\cos 120^\circ + \cos 40^\circ}{2} = \frac{1}{2} \left(\frac{-1}{2} + \cos 40^\circ \right) = \frac{-1}{4} + \frac{\cos 40^\circ}{2}$.
 $184(\cos 20^\circ)(\cos 40^\circ)(\cos 80^\circ) = 184(\cos 20^\circ) \left(\frac{-1}{4} + \frac{\cos 40^\circ}{2} \right)$
 $= -46(\cos 20^\circ) + 92(\cos 20^\circ)(\cos 40^\circ)$

$$\begin{aligned}
&= -46(\cos 20^\circ) + 92 \frac{\cos 60^\circ + \cos 20^\circ}{2} \\
&= -46(\cos 20^\circ) + 46\left(\frac{1}{2} + \cos 20^\circ\right) \\
&= 23
\end{aligned}$$

16. **3** Use the base change formula: $\log_a 512 + \log_a 243 - \log_a 6 = 4 \rightarrow \frac{\log 512}{\log a} + \frac{\log 243}{\log a} - \frac{\log 6}{\log a} = 4 \rightarrow \log 512 + \log 243 - \log 6 = 4 \log a \rightarrow \log \frac{512 \cdot 243}{6} = \log a^4 \rightarrow a^4 = \frac{512 \cdot 243}{6} = 256 \cdot 81 = 2^8 \cdot 3^4 = 2^4 \cdot 2^4 \cdot 3^4 = 12^4 \rightarrow a = 12$. The required sum is 3.
17. **79** Since $\overline{AB} \parallel \overline{CD} \parallel \overline{PM}$, $\triangle RAB \sim \triangle RCD \sim \triangle RPM$. If 2 triangles are similar, the ratio of their areas equals the square of the ratio of their corresponding sides: $\frac{[RAB]}{[RCD]} = \frac{RA^2}{RC^2} = \frac{16}{81}$. Recall that $[RAB]$ means the area of $\triangle RAB$. Since $[RAB] = 16$, $[RCD] = 81 \rightarrow x = [ABDC] = 81 - 16 = 65$. Similarly, $\frac{[RAB]}{[RPM]} = \frac{RA^2}{RP^2} = \frac{16}{225}$ and since $[RAB] = 16$, $[RPM] = 225 \rightarrow y = [CDPM] = 225 - 81 = 144$. Then, $y - x = 144 - 65 = 79$.
18. **53** $6x + 5y = 28$ & $x = 19 + 2z \rightarrow 6(19 + 2z) + 5y = 28 \rightarrow 114 + 12z + 5y = 28 \rightarrow 12z + 5y = -86$. Multiply the latter equation by 2 and the original third equation by 5 and subtract the former from the latter to yield $z = 17$ & $x = 53$.
19. **13** $S = \frac{1}{2} + \left(\frac{1}{3} + \frac{2}{3}\right) + \left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right) + \left(\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5}\right) + \cdots + \left(\frac{1}{13} + \frac{2}{13} + \frac{3}{13} + \cdots + \frac{12}{13}\right)$
 $= \frac{1}{2} + 1 + \frac{3}{2} + 2 + \cdots + 6$
 $= \frac{12}{2} \left(6 + \frac{1}{2}\right)$
 $= 39$
 Finally, $39/3 = 13$.
20. **19** Since $49^x - 7^{x+1} + 12 = 0$ is equivalent to $(7^x)^2 - 7^x + 12 = 0$, let $w = 7^x$, to get the equivalent equation: $w^2 - 7w + 12 = 0 \rightarrow (w - 4)(w - 3) = 0 \rightarrow w = 3$ or $w = 4 \rightarrow 7^x = 4 \rightarrow x = \log_7 4$ or $7^x = 3 \rightarrow x = \log_7 3$. The sum of these solutions is $\log_7 12$ and the required sum is 19.