Grade Level 9 - NMT 2023 **Solutions**

1. **32** $|f(-3)| = |2(-3)^2 - 3(-3) + 5| = |18 + 9 + 5| = 32.$

2. **7** First, $(2x-3)\,(4x^2 - 5x - 6) = 8x^3 - 10x^2 - 12x - 12x^2 + 15x + 18$. The sum of the coefficients of the expanded polynomial is 8 – 10 – 12 – 12 + 15 + 18 = 7.

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3. **9** $A = \frac{1}{2}bh = \frac{1}{2}(3\sqrt{5})h = 30; 3\sqrt{5}\,h = 60; h = \frac{20}{\sqrt{5}} = 4\sqrt{5}; a + b = 4 + 5 = 9.$

4. **6** $5(2) + x(1.50) \leq 20; 1.5x \leq 10; x \leq 6\frac{2}{3}$; maximum of 6 cans of soda

5. **60** $\frac{4}{5}x - 1 = \frac{3}{4}x + 2; \frac{16}{20}x - 1 = \frac{15}{20}x + 2; \frac{1}{20}x = 3; x = 60.$

6. **9** The slope of the line is $\frac{\Delta y}{\Delta x} = \frac{1-11}{-2-3} = -\frac{10}{-5} = 2$. Use one of the points, say (3,11), to solve for the y-intercept. Then $11 = 2(3) + b$ and $b = 5$. Finally, substitute $(2, y)$ into $y = 2x + 5$, resulting in $y = 2(2) + 5 = 9$.

7. **29** The x-coordinate of the vertex is $-\frac{b}{2a} = \frac{14}{2} = 7$. The $y -$ coordinate is $7^2 - 14(7) + 20 = 49 - 98 + 20 = -29$. Finally $|-29| = 29$.

8. **100** $8x + 4y = 12x - 4y; 8y = 4x; x = 2y; 8x + 4y + 9x + 7y = 180; 17x + 11y = 180; 17(2y) + 11y = 180; 45y = 180; y = 4; x = 8; m < AEC = 9x + 7y = 9(8) + 7(4) = 100.$

9. **85** $3\frac{3}{4}$ft $= 45$in; $5\frac{5}{6}$ft $= 70$in; $45 + 16x = 70 + 6x; 10x = 25; x = 2.5$; height of trees $= 45 + 16(2.5) = 45 + 40 = 85$in or $70 + 6(2.5) = 70 + 15 = 85$in.

10. **940** x = original number of cat owners; $(x + 100) - 0.15(x + 100) = x - 56;\ 0.85x + 85 = x - 56; 0.15x = 141; x = 940.$

11. **33** $\frac{ac}{bd} = \frac{60}{100} = \frac{3}{5};\ 5ac = 3bd; ac = \frac{3bd}{5};\ \frac{5ac - bd}{7bd - 4ac} = \frac{5\left(\frac{3bd}{5}\right) - bd}{7bd - 4\left(\frac{3bd}{5}\right)} = \frac{3bd - bd}{7bd - \frac{12}{5}bd} = \frac{2bd}{\frac{23}{5}bd} = \frac{10}{23};\ |p + q| = |10 + 23| = 33.$

12. **9** $2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}} = 2^{\frac{1}{4}}(4) = 2^{\frac{1}{4}}(2^2) = 2^{\frac{9}{4}}; p = 9.$

13. **7** $23^0 = 1, 23^1 = 23, 23^2 = 529, 23^3 = 12167, 23^4 = 278421;$ units digits are 1,3,9,7,1,3,9,7 … ; $\frac{2023}{4} = 505r3$; thus it has the same units digit as 23^3 or 7.

14. **113** There are 6 balls with odd numbers and 5 balls with even numbers. An even sum will contain all even numbered balls or a combination of even numbered balls and an even number of odd numbered balls. (i.e., 4 even and 2 odd, 2 even and 4 odd, 0 even and 6 odd). There are 11C6 or 462 ways to select 6 of the eleven balls. $5C4 \times 6C2 + 5C2 \times 6C4 + 5C0 \times 6C6 = 5(15) + 10(15) + 1 = 226;\ P(\text{even sum}) = \frac{226}{462} = \frac{113}{231}; a = 113.$

15. **210** $A = \frac{h}{2}(39+52) = \frac{91h}{2}$; $h^2 + x^2 = 25$; $h^2 = 25 - x^2$; $h^2 + (13-x)^2 = 144$; $h^2 = -25 + 26x - x^2$; $25 = 13x$; $x = \frac{25}{13}$; $h^2 = 25 - x^2$; $h^2 = 25 - \frac{625}{169} = \frac{25(169)-625}{169} = \frac{25(169-25)}{169} = \frac{25(144)}{169} = h^2$; $h = \frac{5(12)}{13} = \frac{60}{13}$; $A = \frac{91\left(\frac{60}{13}\right)}{2} = 210$.

Grade Level 10 – NMT 2023 **Solutions**

1. **110** The greatest palindrome less than 9999 has the form 9xx9. The greatest value of x that is less than 9 is 8. Therefore, the greatest palindrome that is less than 9999 is 9889. The positive difference between these numbers is 9999 – 9889 $= 110$.

2. **15** The diameter of a wheel and the number of complete rotations of the wheel vary inversely. Therefore, $30 \cdot 10 = 20x \rightarrow x = 15$. The answer is 15.

3. **203** The given expression can be rewritten as $2 - 1 + 4 - 3 + 6 - 5 + \cdots + 406 - 405 = 203$.

4. **119** Since 2023 ends in 3, it is not divisible by 2 or 5. The sum of the digits of 2023 is 7. So, it is not divisible by 3. But, 7 does divide 2023 with zero remainder. The quotient is 289 and $289 = 17^2$. Therefore, the two distinct prime factors are 7 and 17, and the required product is 119.

5. **63** If two lines share more than one point, then they are the same line. Rewrite the given equations as: $y = \frac{2}{7}x - \frac{5}{7}$ and $y = \frac{c}{63} - \frac{a}{63}x$. Then set the appropriate coefficients equal: $\frac{c}{63} = -\frac{5}{7} \rightarrow c = -45$ and $-\frac{a}{63} = \frac{2}{7} \rightarrow a = -18 \rightarrow |a + c| = 63$.

6. **511** The sum of the roots, $r_1 + r_2 = -\frac{b}{a} = \frac{1022}{1}$. Therefore, $\frac{r_1+r_2}{2} = 511$.

7. **200** Since the circular face (labeled 1) and its shadow are similar and in the ratio of 3:10 in their linear dimension, the ratio of their areas is 9:100. Since the area of that circular face is 18, the area of the shadow is 200.

8. **107** Use the fact that the product of the roots of a polynomial equation is the constant term. Therefore $abc = 210 = 2 \times 3 \times 5 \times 7$, and the consecutive integers are 5, 6 and 7. For a cubic equation, the coefficient of the linear term $k = ab + bc + ac = 5 \times 6 + 6 \times 7 + 5 \times 7 = 107$.

9. **167** A sum of eleven can be obtained with a 5 on the first roll and a 6 in the second roll or vice versa. $P(5) = \frac{5}{21}$ and $P(6) = \frac{6}{21}$. Thus, $P(\text{sum of } 11) = 2\left(\frac{5}{21}\right)\left(\frac{6}{21}\right) = \frac{60}{441} = \frac{20}{147}$. So, the required sum $m + n$ is167.

10. **496** The equation of the line connecting the center and the original point can be written as $r(t) =< 1, -5 > \; +t < 2, 3 >$. Then, $r(1) =< 3, -2 >$. So, the image is $r(100) =< 201, 295 >$. Therefore, $a + b = 496$. Alternatively, the equation of $\overleftrightarrow{CP}$ is $y + 5 = \frac{3}{2}(x - 1)$. Then $x' = 1 + 100 \cdot 2 = 201$ and $y' = -5 + 100 \cdot 3 = 295$ and the required sum is 496.

11. **24** Note that $840 = 2^3 \times 3 \times 5 \times 7$. For even factors, each factor should be in the form of $2 \times 2^a \times 3^b \times 5^c \times 7^d$, where a can be 0, 1, or 2 and b, c, and d are either 0 or 1. So there are $3 \times 2 \times 2 \times 2 = 24$ even integer factors. Alternatively, from $840 = 2^3 \times 3 \times 5 \times 7$, there are $4 \cdot 2^3 = 32$ factors. There are 8 factors that are odd: $1, 3, 5, 7, 3 \cdot 5, 3 \cdot 7, 5 \cdot 7$, and $3 \cdot 5 \cdot 7$. Thus, there are $32 - 8 = 24$ even factors.

12. **71** Use the Law of Cosines where angle A is opposite the side whose length is 6: $6^2 = 4^2 + 5^2 - 2(4)(5)\cos C \rightarrow \cos C = \frac{1}{8} \rightarrow \sin C = \frac{\sqrt{63}}{8}$. Then use the extended Law of Sines: $\frac{c}{\sin C} = 2R \rightarrow \frac{6}{\frac{\sqrt{63}}{8}} = 2R \rightarrow R = \frac{24}{\sqrt{63}}$. Then, the area of the circle, $\left(\frac{24}{\sqrt{63}}\right)^2 = \pi\left(\frac{64}{7}\right)$. The required sum is $a + b = 71$.

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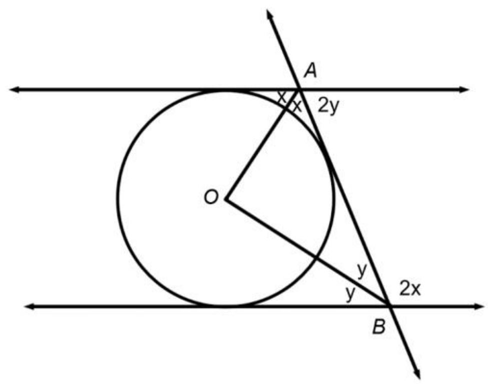
13. **90** In the diagram, draw $\overline{OA}$ and $\overline{OB}$. It can be shown using the two pairs of congruent triangles formed by drawing radii from point O to each of the three points of tangency that both $\overline{OA}$ and $\overline{OB}$ bisect the angles formed by two pairs of the tangent lines. Properties of parallelism yield $2x + 2y = 180 \rightarrow x + y = 90$. In triangle $OAB, m < AOB = 180 - x - y = 180 - 90 = 90$.



14. **502** Since $f(x)$ is a cubic function, there is a constant difference at the third level of finite differences. When the order is reversed, $f(5) = 502$:

x	f(x)			
0	7			
1	18	11		
2	61	43	32	12
3	148	87	44	
4	a			
5	b			

x	f(x)			
0	7			
1	18	11		
2	61	43	32	12
3	148	87	44	12
4	a	143	56	12
5	b	211	68	

a = 291 b = 502

Alternatively, from the given, together with $f(x) = ax^3 + bx^2 + cx + d$, $a + b + c + 7 = 18$, $8a + 4b + 2c + 7 = 61$, and $27a + 9b + 3c + 7 = 148$. Eliminate the $a \rightarrow 2b + 3c = 17$ and $3b + 4c = 26 \rightarrow (a, b, c) = (2, 10, -1) \rightarrow f(5) = 502$.

15. **6** Rewrite the given equation as $\log_x 2 + \log_x 3 + \log_x 6 = \log_x(2 \cdot 3 \cdot 6) = 2 \rightarrow x^2 = 36 \rightarrow x = 6$.

Grade Level 11 – NMT 2023 **Solutions**

1. **34** If x is the number of heads on the fifth day, then the average is $\frac{72+43+61+40+x}{5} = 50 \rightarrow 216 + x = 250 \rightarrow x = 34$.

2. **28** The slope of $f(x) = \frac{f(10)-f(4)}{10-4} = \frac{24}{6} = 4$. Then, $\frac{f(8)-f(1)}{8-1} = 4 \rightarrow \frac{f(8)-f(1)}{7} = 4 \rightarrow f(8) - f(1) = 28$.

3. **72** Rewrite the given equation as $3^{4\sqrt{x}} = 3^{1/2} \rightarrow 4\sqrt{x} = \frac{1}{2} \rightarrow \sqrt{x} = \frac{1}{8} \rightarrow \frac{9}{\sqrt{x}} = \frac{9}{1/8} = 72$.

4. **18** First, $g(5) = (5)^2 + 3(5) = 40 \rightarrow f(x) = 2x + 4 = 40 \rightarrow x = 18 \rightarrow f^{-1}(40) = 18$.

5. **25** The solutions must be integral, so inspection quickly yields $x = 9$ or 16 and the required sum is 25. Or: square the given equation: $\left(\sqrt{25-x} + \sqrt{x}\right)^2 = 7^2 \rightarrow (25 - x) + 2\sqrt{25x - x^2} + x = 49 \rightarrow 2\sqrt{25x - x^2} = 24 \rightarrow \sqrt{25x - x^2} = 12$. Square again: $\left(\sqrt{25x - x^2}\right)^2 = 12^2 \rightarrow 25x - x^2 = 144 \rightarrow 0 = x^2 - 25x + 144 \rightarrow (x - 9)(x - 16) = 0 \rightarrow x = 9$ or 16. Of course, the sum is 25.

6. **400** Use co-functions to rewrite the given expression as $100\left(\frac{\sin^2(55^o)+\cos^2(55^o)}{\sin^2(30^o)}\right) = 100\left(\frac{1}{\left(\frac{1}{2}\right)^2}\right) = 100\left(\frac{1}{1/4}\right) = 100(4) = 400$.

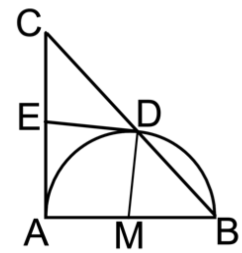
7. **8** The sum of the roots of $x^2 + 5x + m = 0$ is -5 and the product of the roots is m. The sum of the roots of $x^2 + nx + 5 = 0$ is $-n$ and the product is 5. Since the roots of the first quadratic are double the roots of the second, $-5 = 2(-n)$ and $m = 4(5)$. Therefore, $n = 5/2$ and $m = 20$ so $m/n = \frac{20}{5/2} = 8$.

8. **289** Since the given quadratic equation has only one solution, the discriminant must be equal to 0. So, $= (5 + 3i)^2 - 4(1/2)(c) = 0 \rightarrow 25 + 30i + 9i^2 - 2c = 0 \rightarrow 25 + 30i + 9(-1) = 2c \rightarrow 16 + 30i = 2c \rightarrow 8 + 15i = c$. The required product is $(8 + 15i)(8 - 15i) = 64 - 225i^2 = 64 + 225 = 289$

9. **351** Start with $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \rightarrow (x + y)^3 = x^3 + y^3 + 3xy(x + y) \rightarrow (9)^3 = x^3 + y^3 + 3(14)(9) \rightarrow 729 = x^3 + y^3 + 378 \rightarrow x^3 + y^3 = 351$

10. **49** An equation of the parabola (in vertex form) is $y - 4 = a(x - 1)^2$. Since the point with coordinates $(3, 24)$ is on the parabola, $24 - 4 = a(3 - 1)^2 \rightarrow 20 = 4a \rightarrow a = 5$. Since the point with coordinates $(-2, k)$ is on the parabola, $k - 4 = 5(-2 - 1)^2 \rightarrow k - 4 = 5(-3)^2 \rightarrow k - 4 = 45 \rightarrow k = 49$

11. **405** As shown at right, when a radius is drawn from point D to the midpoint M of $\overline{AB}$, triangle DMB is isosceles with $\angle MDB \cong \angle DBM$. Note that $\angle EDC$ and $\angle ECD$ are complementary to the previous two angles. That makes triangle CED isosceles. Then $DE = CE = AE = 3$. Using the theorem about a tangent and a secant to a circle: $CA^2 = CD \times CB \rightarrow 6^2 = 4(4 + x) \rightarrow x = 5$. Use the Pythagorean theorem to get $AB = \sqrt{45}$. Then, the area of the triangle is $k = \frac{1}{2}(6)\sqrt{45}$. Therefore $k^2 = 405$.



12. **12** Of the 25 consecutive even integers, the smallest is x. Then the sum of the sequence is $\frac{25(x+(x+48))}{2} = 900 \rightarrow 25(2x + 48) = 1800 \rightarrow 2x + 48 = 72 \rightarrow x = 12$.

13. **21** Since $\overline{EF}$ is a midsegment of $\Delta ABC, AE = EB = 13,\ BF = FC = 20$ and the length of an altitude from vertex B to $\overline{EF}$ is 12. If the foot of that altitude is point G, then by the Pythagorean Theorem in ΔBGE and $\Delta BGF,\ EG = 5$ and $GF = 16$. Thus, $EF = 21$.

8. **32** Write $\log_{16} x + \log_x 16 = \frac{\log x}{\log 16} + \frac{\log 16}{\log x} = y$. Multiply by$(\log x)(\log 16)$ to get $(\log x)^2 - y(\log 16)(\log x) + (\log 16)^2 = 0$. To have $\log x \in \mathbb{R}, y \geq 2$. Then, $\log_{16} x + \log_x 16 = 2$ has a solution of $x = 16$. Therefore, $a = 16, b = 2$. So, $ab = 32$.

Alternatively, the sum of the reciprocals is a minimum when the term and its reciprocal are equal. Change to base 4: $y = \frac{\log_4 x}{2} + \frac{2}{\log_4 x} \rightarrow \frac{\log_4 x}{2} = \frac{2}{\log_4 x} \rightarrow \log_4 x = 2 \rightarrow x = 16, y = 2$ and the required product is 32.

9. **462** First, distribute one cookie to each friend. That leaves six more cookies to distribute to six friends. Use the stars and bars theorem. There are 6 stars and 5 bars (6 friends – 1). Therefore, $C(6 + 5, 5) = \frac{11!}{6!5!} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 462$ gives the required number of ways to complete the distribution. Consult: Wikipedia stars and bars (combinatorics).

Grade Level 12 – NMT 2023 Solutions

1. **72** In all cases, x% of y is $\frac{x}{100}y = \frac{xy}{100}$ and y% of x is $\frac{y}{100}x = \frac{xy}{100}$, which is the exact same value.

2. **25** Since it forms a square, $n = c^2$, where c is an integer. Reforming those c^2 tiles into two other squares means that $c^2 = a^2 + b^2$, where all variables are integers. This means that (a, b, c) is a Pythagorean triple, the smallest of which is $(3, 4, 5)$. So there are a total of $5^2 = 25$ tiles.

3. **144** Let $n = 0$. Thus, $f(0) = 1$ and $M = 1$. So, $(12M)^2 = 144$.

4. **10** Since $d = vt$ and $d = (v+12)(t-5)$, we find that $vt = vt + 12t - 5v - 60 \rightarrow 12t = 5v - 60 \rightarrow t = \frac{5}{12}v + 5$. The smallest positive integer v for which t is an integer is $v = 12 \rightarrow t = 10$ and $d = 120$.

5. **47** The sequence consists of all the prime numbers, in order. Formal proof of this would be inductive, but writing out the first few terms should convince you of this. The 15th prime number is 47.

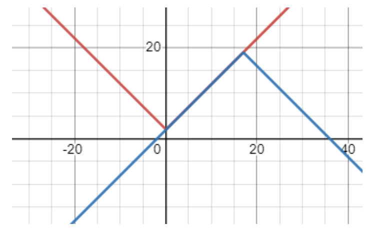
6. **8** Method 1: $(1+i)^2 = 2i \rightarrow (1+i)^3 = 2i(1+i) = -2+2i$

$\frac{16}{-2+2i} \cdot \frac{-2-2i}{-2-2i} = \frac{16(-2-2i)}{8} = 2(-2-2i) = -4-4i$, so $|-4-4| = 8$.

Method 2: [Note: $\text{cis}\theta = \cos\theta + i\sin\theta$] Using polar form:

$\frac{16}{(1+i)^3} = \frac{16\text{cis } 360°}{(\sqrt{2}\text{cis } 45°)^3} = \frac{16\text{cis } 360°}{2\sqrt{2}\text{cis } 135°} = 4\sqrt{2}\text{cis}225° = 4\sqrt{2} = \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -4 - 4i.$

7. **17** The graph of $y = |x| + 2$ is an upward-opening V-shaped graph with vertex at $(0,2)$, as shown. The graph of $y = -|x-h| + (36-h)$ is a downward-opening V-shaped graph with vertex at $(h, 36-h)$. In order to have an infinite number of solutions, a portion of the graphs must overlap, as shown. This will occur only if the vertex of the second graph lies on the first graph. Substituting $(h, 36-h)$ into the first equation yields $36 - h = |h| + 2 \rightarrow |h| + h = 34$. If $h < 0$, we get $0 = 34$ (no solutions), so $h \geq 0$ and $2h = 34 \rightarrow h = 17$.

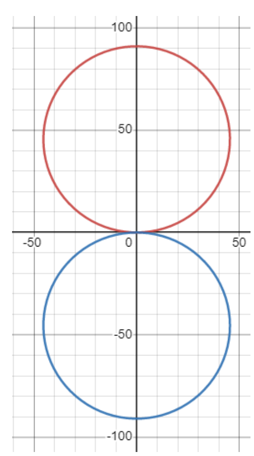


8. **2** Let the side length of the square be x. The removal of the square region adds the two vertical sides of the square to the perimeter of the heptagon, increasing the heptagon's perimeter by $2x$. The area of the heptagonal region decreases by the square's area, which is x^2. So, $x^2 = 2x \rightarrow x = 2$. (Note: the fact that the region is heptagonal is irrelevant.)

9. **168** Since the area of the triangle is 42 and the length of the hypotenuse is 14, the length h of the altitude to the hypotenuse can be found: $\frac{1}{2}(14)h = 42 \rightarrow h = 6$. The foot of this altitude divides the hypotenuse into two segments of lengths a and $14-a$. The solid of revolution is two cones sharing a common base, with a combined volume of

$\frac{1}{3}\pi(6)^2 a + \frac{1}{3}\pi(6)^2(14-a) = 12\pi a + 168\pi - 12\pi a = 168\pi$ and $k = 168$.

10. **182** If $0° \leq \theta \leq 180°$, $\sin\theta \geq 0$, so $r = 91\sin\theta$, which is the upper circle in the graph shown. In quadrants III and IV, r is still positive, so another circle congruent to the upper circle is drawn in those two quadrants. Each circle has diameter of 91, so the total circumference of both circles is $2\pi \cdot 91 = 182\pi$ and $k = 182$.

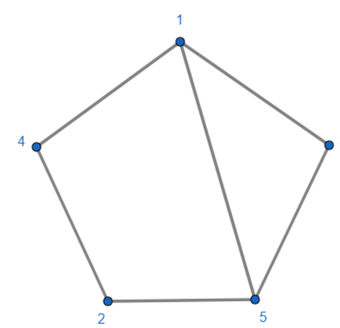


11. **899** Let the side length of the cube be x. Then $V = x^3$ and $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$. Since the surface area is $A = 6x^2$, $6x^2 = 62 \rightarrow 3x^2 = 31$. Plugging in, $\frac{dV}{dt} = (31)(29) = (30+1)(30-1) = 30^2 - 1^2 = 899$.

12. **88** Since $y = 87$ is horizontal $\frac{dy}{dx} = 0$ at two points where $y = 87$,
$\frac{dy}{dx} = 4x^3 + 2bx = 0 \rightarrow 2x(2x^2 + b) = 0 \rightarrow x^2 = -\frac{b}{2}$. Plugging into the original function,
$y = (x^2)^2 + bx^2 + 2023 \rightarrow 87 = \frac{b^2}{4} - \frac{b^2}{2} + 2023 \rightarrow \frac{b^2}{4} = 1936 \rightarrow b^2 = 1936 \cdot 4 \rightarrow b = \pm 44 \cdot 2 = \pm 88$.
(Note: $x^2 = -\frac{b}{2}$ implies that $b < 0$.) So, $|b| = 88$.
Alternate Method: The 4th degree function can be translated down so that it is tangent to the x-axis in two places. Since the function is even, $y = x^4 + bx^2 + 1936$ must be a perfect square. Thus, $b = \pm 2(44) = \pm 88$ and $|b| = 88$.

13. **777** Note that $f'(x) = \lim_{a \to o} \frac{f(x+a) - f(x)}{a} = \lim_{a \to o} \frac{3x^2 a + 3xa^2 + a^3 + 9a}{a}$
$= \lim_{a \to o} (3x^2 + 3xa + a^2 + 9) = 3x^2 + 9$, so $f'(16) = 3(16)^2 + 9 = 777$.

14. **14** Consider the graph at right. All vertices that differ by more than 1 are connected by edges. The problem now becomes one of finding the number of paths through the graph that use each vertex exactly once (these are known as Hamiltonian paths). Let's count by cases.



If the edge between 1 and 5 is not used, then we can start at any vertex and proceed in either direction around the circuit, for a total of 10 paths.
If the edge between 1 and 5 must be used, we now consider various starting points:

- If we start at 1 or 5, there are no possible paths, since we must start by moving between 1 and 5 and cannot reach both {2, 4} and 3.
- If we start from 2 or 4, there is one path each (24153 and 42513).
- If we start from 3, there are 2 paths (31524, 35142)

This gives us a total of 14 paths.

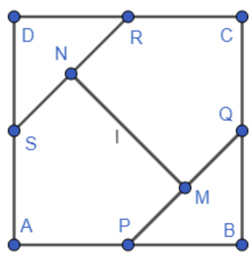
15. **864** Method 1: Use L'Hôpital's rule: $\lim_{h \to 0} \frac{4(6+h)^3 - 4(6-h)^3(-1)}{2} = \frac{4(6)^3 + 4(6)^3}{2} = 4(6)^3 = 864$.
Method 2: Recognize that $\frac{f(x+h) - f(x-h)}{2h}$ is the slope of the secant line centered at x, from a point h units to the left of x to a point h units to the right of x. As h approaches zero, the secant line will approach the tangent line and so this limit is just another expression for $f'(x)$. (It is known as the symmetric derivative.) So, the limit equals $f'(6)$, where $f(x) = x^4$.

1. **20** If the Yorkie's weight is x pounds, then the Beagle's weight is $2x + 2$ pounds and the Lab's weight is $(2x + 2) + 11 = 2x + 13$ pounds. Since the average weight of the three dogs is 20 pounds, the sum of their weights must be 60 pounds. So, $x + 2x + 2 + 2x + 13 = 60 \rightarrow x = 9 \rightarrow 2x + 2 = 20$.

2. **96** If each side is multiplied by a factor of 1.4, then the area is multiplied by a factor of $1.4^2 = 1.96$. Thus, $k = 96$.

3. **24** $16^{\frac{23}{20}} = \left(16^{\frac{1}{20}}\right)^{23} = \left(16^{\frac{1}{4}}\right)^{\frac{23}{5}} = (2)^{\frac{23}{5}} = (2^4)2^{\frac{3}{5}} = 16\sqrt[5]{8}$. The minimum possible value is $a + b = 16 + 8 = 24$.

4. **28** From the diagram below, we see that $BP = BQ = 20, PQ = 20\sqrt{2}$ and $B = MQ = 10\sqrt{2}$. The distance we seek is $NM = 40\sqrt{2} - 20\sqrt{2} = 20\sqrt{2} \approx 28$. Alternatively, since quadrilateral $PQRS$ is also a square, the required distance is equal to the length of any side of the square.

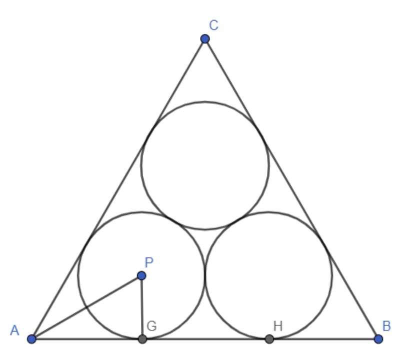


5. **4** First, $f(2) = |2 + 3| = 5$. Next, $f(5) = 2(5) - 12 = -2$. Then, $f(-2) = |-2 + 3| = 1$. Finally, $f(1) = |1 + 3| = 4$.

6. **833** Since $2023 = 7 \cdot 17^2$ and $2023N = 7^3 \cdot 17^3$, then $N = 7^2 \cdot 17 = 833$.

7. **938** In the given geometric sequence $a_1 = 2$ and $a_{10} = 2023$. Using $a_n = a_1 r^{n-1}$, $2r^9 = 2023 \rightarrow r \approx 2.157117 \rightarrow a_9 = 2023/r \approx 938$.

8. **852** Note that the line drawn from vertex A to the center of the nearest circle is an angle bisector, creating a 30°-60°-90° triangle as shown in the figure. Label $PG = r$, the radius of circle P. We have $AG = r\sqrt{3}$ and $GH = 2r \rightarrow AB = 2r + 2r\sqrt{3} = 90 \rightarrow r = \frac{45}{1+\sqrt{3}}$ and $\pi r^2 \approx 852$.



9. **945** We seek the probability that each student is paired with his or her best friend. Any one of the ten girls has a $\frac{1}{9}$ chance of being paired with her best friend. Assuming that happens, eight girls remain. Any one of these girls has a $\frac{1}{7}$ chance of being paired with her best friend. This process continues. Thus, the probability that each girl is paired with her best friend is $\frac{1}{9}\cdot\frac{1}{7}\cdot\frac{1}{5}\cdot\frac{1}{3}\cdot\frac{1}{1}=\frac{1}{945}$. Similarly, the probability that each boy is paired with his best friend is also $\frac{1}{945}$. Therefore, the required probability is $\frac{1}{945^2}$ and $n=945$.

10. **936** **Method 1**: The line $17x+119y=2023$ has intercepts at (0,17) and (119,0) with lattice points located in between every 7 units along the $x-$axis: (0,17), (7,16), (14,15), (21,14), ... (112,1), (119,0). We can count the first quadrant lattice points under the line one row at a time:

- $y=16$: 6 points: (1,16), (2,16), (3,16), ... (6,16)
- $y=15$: 13 points: (1,16), (2,16), (3,16), ... (13,16)
- $y=14$: 20 points, etc.
- ...
- $y=1$: 111 points.

Summing these we have $6+13+20+\ldots+111=(6+111)(16/2)=936$

Method 2: Pick's theorem gives the area inside the triangle formed by the line and the x and y axes to be A = I + ½B – 1 where I is the number of interior lattice points and B is the number of points on the border (edges) of the triangle. There are 18 lattice points on the $y-$axis, including the origin, 119 additional points on the $x-$axis, and 16 additional lattice points on the given line. Thus, B = 18 + 119 + 16 = 153 and A = ½ (17)(119) → I = ½(2023) – ½(153) + 1 = 936.

Team Problem Solving - NMT 2023 **Solutions**

1. **120** **Method 1:** Create a table of values for h, and possible tu numbers.

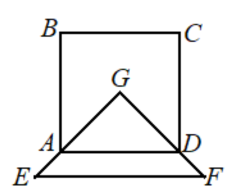
h	1	2	3	4	5	6	7	8	9
additional *tu*	–	10	21, 20	32, 31, 30	43, 42, 41, 40	54, 53, 52, 51, 50	65, 64, 63, 62, 61, 60	76, 75, 74, 73, 72, 71, 70	87, 86, 85, 84, 83, 82, 81, 80
Total	0	1	2 + 1 = 3	3 + 3 = 6	6 + 4 = 10	10 + 5 = 15	15 + 6 = 21	21 + 7 = 28	28 + 8 = 36

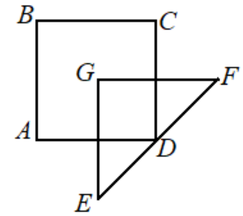
Each Total shown gives the count of three-digit numbers starting with the value of h shown at the top of the column. (The "+1" in the $h=3$ column indicates that the tu of 10 from the prior column is added to the tu values of 21 and 20, and so on for each column.) Sum these Totals to find the overall Total, 120.

Method 2: Given 10 available digits, there are $10C3$ ways of choosing any three of them. The digits of each of these can be ordered from greatest to least in only one way. Thus, the total number of 3-digit numbers that satisfy the requirement that $h>t>u$ is $10C3=120$.

2. **19** Consider the perfect squares from 1 to 12. Find the three perfect squares that sum to $13^2=169$. There is only one set that has a sum of 169: $9+16+144$. The requested sum is $3+4+12=19$.

3. **36** Draw the triangle so that it contains two vertices of the square (see diagram). It is clear from the diagram that the area of ΔGAD equals ¼ the area of the square, or $\frac{12^2}{4} = 36$. Also, the area of $\Delta GEF = \frac{1}{2} \cdot 12^2 = 72$. So, the required area of trapezoidal region $AEFD = 72 - 36 = 36$. Note that the required region is interior to ΔGEF and outside the square. Another diagram that makes this clear can be seen in the second figure. An interesting fact is that rotating the triangle from the first position to the second position maintains the area of the region inside the square throughout all intermediate positions.





4. **142** Since $\sqrt{1000000} = 1000$, we want to know how many multiples of 7 are less that 1000. Divide 1000 by 7 to get 142 with a remainder of 6. There are 142 multiples of 7 such that their squares are less than 1,000,000 (7, 14, 21, ... $7 \times 142 = 994$).

5. **650** Replace the five E's with the digit 8. The letter T must be 1, the maximum carry over from the ten-thousands column. Since the two 8's in the thousands column add up to 18, V + V + S must add to at least 20. Let V = 7 and S = 6. Now realize that 8 + 8 + I must sum to 21, so I = 5 and there can be no carry over from the ones column. Since the sum in the tens column is 21, the hundreds column has a sum of 22 so N = 2 forcing X = 0 and W = 3. The number SIX is 650.

S 8 V 8 N	6 8 7 8 N	6 8 7 8 2
S 8 V 8 N	6 8 7 8 N	6 8 7 8 2
+ S I X	+ 6 5 X	+ 6 5 0
1 W 8 N 1 Y	1 W 8 N 1 Y	1 3 8 2 1 4

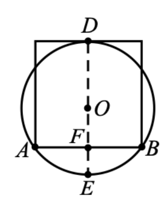
6. **45** Examine the table of starting number positions: Notice that the starting position of number n is 1 more than the sum of the numbers from 1 to $n - 1$. To calculate the position of the first n numbers use $S = 1 + (n-1)(n)/2$. For example, the first number 6 appears in position $1 + 5(6)/2 = 16$. Use this idea to find n when the position is 1000: Solve for n in the inequality $1 + (n-1)(n)/2 \leq 1000 \rightarrow (n-1)(n) \leq 1998 \rightarrow 44 \times 45 = 1990$ so $n = 45$. The number 45 starts in the position $1 + (44)(45)/2 = 991$ and continues beyond position 1000.

n	1	2	3	4	5	6	…
Position	1	2	4	7	11	16	…

7. **20** Count the number of squares of each size. The total number of squares is 9 + 4 + 1 + 1 + 2 = 20.

Size	1 × 1	2 × 2	3 × 3	$\sqrt{2} \times \sqrt{2}$	$\sqrt{5} \times \sqrt{5}$
Number	9	4	1	4	2

8. **50** Draw diameter $\overline{DE}$ from the point of tangency of the square perpendicular to $\overline{AB}$. Apply the theorem $DF \times FE = AF \times FB$ so $40 \times FE = 20 \times 20$ and $FE = 10$. The diameter of the circle is $40 + 10 = 50$ and the circumference is 50π so $n = 50$.



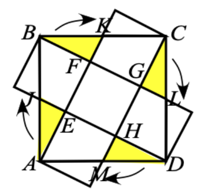
9. **135** There can be only two elements of 17 since the middle number of the five is 85. The mean is 68, so the sum of the five numbers is $5 \times 68 = 340$. The sum of the three least numbers is 17 + 17 + 85 = 119 so the remaining two numbers have a sum of 340 – 119 = 221. The minimum for the fourth number is 86 so the greatest value for the fifth number is 221 – 86 = 135.

10. **42** Solve by inverting all three fractions and changing the direction of the inequalities: $\frac{5}{4} < \frac{12}{n} < \frac{9}{2}$ $\rightarrow \frac{2}{9} < \frac{n}{12} < \frac{4}{5}$. Multiply by 12 to get $\frac{8}{3} < n < \frac{48}{5} \rightarrow 2\frac{2}{3} < n < 9\frac{3}{5}$. The solution consists of 3, 4, 5, 6, 7, 8, and 9 and the sum is 42. Alternate solution: Rewrite the given inequality with a common numerator: $\frac{540}{432} < \frac{540}{45n} < \frac{540}{120}$. Then $120 < 45n < 432$. The rest of the solution is the same.

11. **88** Let $P(x) = x^5 + x^4 - 30x^3 - 76x^2 + ax + b$. Since $(x+2)$ and $(x-1)$ are factors of $P(x)$, $P(-2) = P(1) = 0$. Hence, $P(-2) = -32 + 16 + 240 - 304 - 2a + b = 0$ and $P(1) = 1 + 1 - 30 - 76 + a + b = 0$. These equations simplify to $-2a + b = 80$ and $a + b = 104$. Solve this system to get $a = 8$ and $b = 96$ so $b - a = 96 - 8 = 88$.

12. **16** Since $f(x) = \frac{3}{2-x}$, 2 is not in the domain of f. $f(f(x)) = \frac{3}{2 - \frac{3}{2-x}} = \frac{6-3x}{1-2x}$. So $\frac{1}{2}$ is not in the domain of $f(f(x))$. $f(f(f(x))) = \frac{3}{2 - \frac{6-3x}{1-2x}} = \frac{6x-3}{x+4}$. So −4 is not in the domain of $f(f(f(x)))$.
The three numbers that are not in the domain of $f(f(f(x)))$ are 2, 1/2, and −4. The product of their squares is $4 \times 1/4 \times 16 = 16$.

13. **80** The area of square $EFGH$ is 1/5 the area of square $ABCD$ which is $20^2 = 400$. This can be seen by rotating each of the smallest right triangles as shown. This results in 4 additional squares each congruent to the square $EFGH$.



14. **134** By observation, one can see that one of the intersection points will be (1, 5). Keep in mind that all answers in this contest are integers from 0 to 999. So, the only possible values for x are 2, 3, and 4. Testing these values one soon finds that the other intersection point has coordinates (3, 125). The sum of the four coordinates is $1 + 5 + 3 + 125 = 134$.

15. **18** There are $9 \times 10 \times 10 \times 10 = 9000$ 4-digit numbers. There are $9 \times 9 \times 8 \times 7 = 4536$ 4-digit numbers such that no digits repeat. Thus, there are $9000 - 4536 = 4464$ 4-digit numbers with at least two digits the same. The sum of the digits is $4 + 4 + 6 + 4 = 18$.

16. **58** In the equation $d + e + f + g = 45$, replace e with $d + 4$, f with $(d + 7)/3$, and g with $5(d - 4)$. Solve the equation: $d + (d+4) + \frac{(d+7)}{3} + 5(d-4) = 45$ to get $d = 8$. Substitute that back to find that $e = 12$, $f = 5$, and $g = 20$. Hence, $4d + 3e + 2f - g = 4(8) + 3(12) + 2(5) - 20 = 58$.

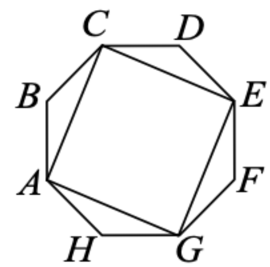
17. **15** First, simplify the radicand: $\frac{-18i}{\frac{1}{12+9i} - \frac{1}{12-9i}} = \frac{-18i}{\frac{1}{12+9i} - \frac{1}{12-9i}} \cdot \frac{(12+9i)(12-9i)}{(12+9i)(12-9i)} = \frac{-18i(144+81)}{(12-9i)-(12+9i)} = 225$.

Finally, take the square root of 225 which equals 15.

18. **168** Each face of the cube has an area of $5 \times 5 - 3 \times 3 = 16$. Since there are 6 faces on a cube, the total outside area is $6 \times 16 = 96$. The surface of each of the 6 holes is comprised of four 1×3 rectangles for a total area of $6 \times 4 \times 1 \times 3 = 72$. The total surface area is $96 + 72 = 168$.

19. **$A_{19} = 4$** The three prime factors of 126 are 2, 3, and 7. Evaluate $\log\left(2^{10} - 8(3)\right) + \log(7 + 3) = \log(1000) + \log(10) = 3 + 1 = 4$.

20. **50** The angles in a regular octagon have a measure of 135°. The side measure in this octagon is $A_{19} = 4$. Let a side of the square be x so its area is x^2. Apply the law of cosines to one of the small triangles, say ΔABC:



$x^2 = 4^2 + 4^2 - 2(4)(4)\cos(135°) = 16 + 16 + 32\cos(45°) = 32 + 16\sqrt{2}$. Thus, $p = 32$, $q = 16$, and $r = 2$ so $p + q + r = 50$.