Grade Level 9 - NMT 2022 **Solutions**

1. **21** $(5x+2)(7x-4) = 35x^2 - 20x + 14x - 8 = 35x^2 - 6x - 8$; $a + b + c = 35 - 6 - 8 = 21$.

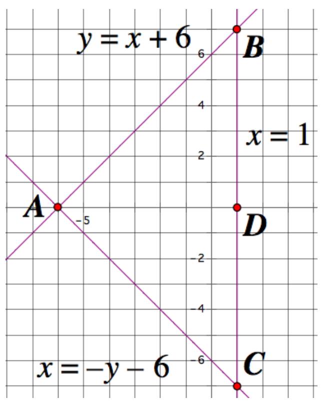
2. **3** $2x^2 - 4x - 6 = 2(x-3)(x+1) = 0 \rightarrow x = 3$ or $x = -1$. The required answer is 3.

3. **18** $3\sqrt{72} - \frac{2}{5}\sqrt{50} = 3(\sqrt{36}\sqrt{2}) - \frac{2}{5}(\sqrt{25}\sqrt{2}) = 3(6\sqrt{2}) - \frac{2}{5}(5\sqrt{2}) = 18\sqrt{2} - \frac{2}{5}(5\sqrt{2})$
$= 18\sqrt{2} - 2(\sqrt{2}) = 16\sqrt{2}$. Therefore, $a + b = 16 + 2 = 18$.

4. **60** $m\angle EBD = 90 \rightarrow m\angle DBC + m\angle ABE = 90$. Since $m\angle DBC = 2m\angle ABE \rightarrow x + 2x = 90 \rightarrow$
$x = 30 \rightarrow m\angle DBC = 60$.

5. **0** $a^b - b^c + c^a = (-1)^2 - (2)^1 + (1)^{-1} = 1 - 2 + 1 = 0$.

6. **49** $\triangle ABC$ is formed. Base $BC = 14$, altitude $AD = 7$. Area $= {}^1/_2\, bh = {}^1/_2\,(14)(7) = 49$.



7. **3** $y^4 - 4y^2 + 3 = 0 \rightarrow (y^2-3)(y^2-1) = 0 \rightarrow (y-\sqrt{3})(y+\sqrt{3})(y-1)(y+1) = 0 \rightarrow$
$y = \pm\sqrt{3}, \pm 1$ and the product $(\sqrt{3})(-\sqrt{3})(1)(-1) = 3$.

8. **0** The units digit of the product is the units digit of the product of the digits 1 through 9. Included in the product of these digits is a factor of 10 (2×5). Therefore, the units digit is zero.

9. **26** The tee shirts from the home team represent 1/3(2/5)=2/15 of the total participants. The tee shirts from the visiting team represent 2/9(3/5)= 2/15 of the total participants. So 4/15 of the participants wore tee shirts and 11/15 wore sweatshirts, and $a + b$ = 11+15 = 26.

   Alternate Solution: Without loss of generality, suppose there are 45 total mathletes (a nice choice because it is divisible by 5, 3, and 9.) Since two-fifths of the total participants are the home team, there are 18 participants on the home team. One-third of them wore tees. So the home team has 6 tees and 12 sweats. Since there are 27 participants on the visiting team and two-ninths of them wore tees, there are 6 tees and 21 sweats on the visiting team. So there are 33 sweats in total out of 45 participants. In simplest terms this is 11/15 so $a + b =$ 11+15 = 26

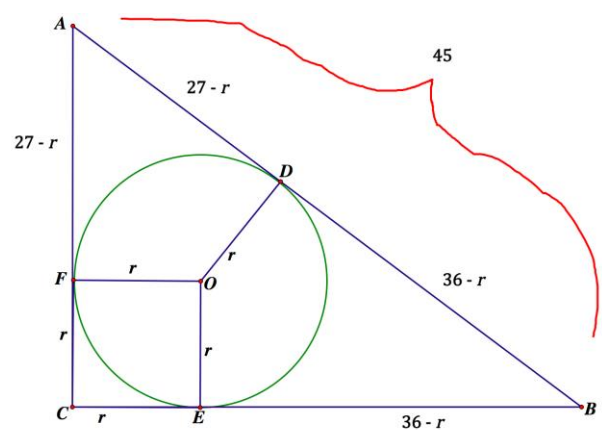
10. **10** The only digits that can be used are 6, 7, 8, 9. There are 4 three-digit numbers beginning with 9 {996,987,978,969}. There are 3 three-digit numbers beginning with 8 {897,888,879}, 2 three-digit numbers beginning with 7 {798,789} and 1 beginning with 6 {699}. Thus there are 10 such numbers, {996,987,978,969,897,888,879,798,789,699}.

11. **27** The mean (x) is 14. The median (y) is 15. The mode (z) is 17 and the range (r) is 7. Substitute these values into the given equation and $k = 27$.

12. **4** There are 10! ways for 10 people to stand in a line. Since the two girls are to be together, treat them a unit. This leaves you 9! ways to arrange 10 people and two ways to arrange the two girls. Therefore, the probability the girls will be next to each other is $\frac{9!(2)}{10!} = \frac{2}{10} = \frac{1}{5}$. Therefore, $b - a = 5 - 1 = 4$.

13. **6** The given expression can be factored into three consecutive integers this way: $x^3 - x = x(x^2 - 1) = (x-1)x(x+1)$. The product of any two consecutive integers must be divisible by 2. The product of three consecutive integers must be divisible by 3. Therefore, the given expression must be divisible by 2 and 3 (and 1). Therefore, the largest integer which divides $x^3 - x$ is 6. Note that among the three consecutive integers, it is possible that only one is divisible by two (if the other two are odd) and that it is never possible to have more than one of the three consecutive integers be divisible by three.

14. **9** The hypotenuse is 45 (3,4,5 triplet × 9). In the diagram, circle O is inscribed in right $\triangle ABC$ with points of tangency D, E and F on $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ respectively. Note that the quadrilateral $OFCE$ is a square. Therefore, $BE = BD = 36 - r$ and $AF = AD = 27 - r$. So $27 - r + 36 - r = 45$ and $r = 9$.



15. **200** $T = \frac{D}{R}$; Bob's time when Art crosses the finish line is $\frac{d-40}{R(Bob)}$ and Carl's time when Art crosses the finish line is $\frac{d-56}{R(Carl)} \rightarrow \frac{d-40}{R(Bob)} = \frac{d-56}{R(Carl)} \rightarrow \frac{R(Bob)}{R(Carl)} = \frac{d-40}{d-56}$; Bob's time: $\frac{d}{R(Bob)}$; Carl's time when Bob crosses the finish line is $\frac{d-20}{R(Carl)} \rightarrow \frac{d}{R(Bob)} = \frac{d-20}{R(Carl)} \rightarrow \frac{R(Bob)}{R(Carl)} = \frac{d}{d-20} \rightarrow \frac{d-40}{d-56} = \frac{d}{d-20} \rightarrow$
$d^2 - 60d + 800 = d^2 - 56d \rightarrow 800 = 4d \rightarrow d = 200$.

Grade Level 10 – NMT 2022 **Solutions**

1. **6** The remaining WiFi strength after it goes through 4 walls will be $\left(\frac{1}{2}\right)^4 = \frac{1}{16} = 0.0625 \approx 6\%$. The required integer answer is 6.

2. **3** In order for a quadratic equation to have more than two solutions, there must be an infinite number of solutions. The equation must be an identity in x. Set the coefficients of the like terms equal to get $a = 2, b = -1$ and $c = 3$.

3. **11** Since both the length and width of the rectangle must be positive integers that sum to 12, half the perimeter, the problem can be solved by trial and error. The dimensions are 11 and 1 and the minimum area is 11.

4. **31** $1798 = 2 \cdot (899) = 2 \cdot (900 - 1) = 2 \cdot (30^2 - 1^2) = 2 \cdot (30 + 1)(30 - 1) = 2 \cdot 31 \cdot 29$
The greatest of the three prime factors of 1798 is 31.

5. **25** In one game, there are 40 minutes of playing time for a team's 5 players. So there are 200 minutes of playing time to be shared equally by 8 players on a team. The result is 25 minutes per player per game.

6. **192** $S = 1 + 11 + 21 + \cdots + 191 = \frac{20}{2}(1 + 191) = 1920 \rightarrow \frac{S}{10} = 192$

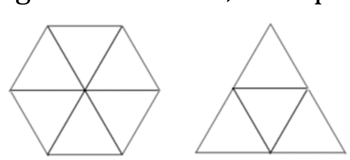
7. **216** The given expression can be re-written as $(4x^2 - 12x + 9)^2$. Then, when it is expanded, the coefficient of x^2 is $4 \cdot 9 + 4 \cdot 9 + 12 \cdot 12 = 216$. Alternatively, we can expand the original binomial and $cx^2 = \binom{4}{2}(2x)^2(-3)^2 \rightarrow c = 216$.

8. **4** By the definition of absolute value we have the following:

$$x + 13 = x - 21 \text{ or } x + 13 = -(x - 21)$$

The first leads to a false statement so it has no solution. The second gives the solution $x = 4$.

9. **10** Partition the original equilateral triangle and regular hexagon into smaller equilateral triangles: 4 in the original equilateral triangle and 6 in the regular hexagon. Because the ratio of the areas of the original equilateral triangle to the regular hexagon is 2:3 (i.e., 4:6), the smaller equilateral triangles are all congruent to each other. The perimeters of the original equilateral triangle and the regular hexagon are each equal to the sum of the lengths of 6 sides of the smaller equilateral triangles. Therefore, their perimeters are each equal to 10.

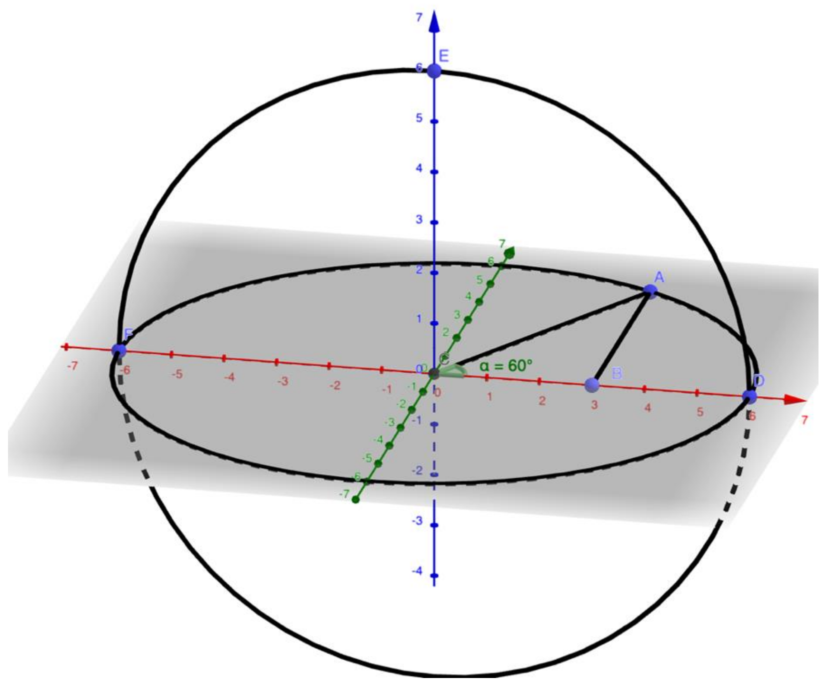


10. **0** The given line makes a 45-degree angle with the x–axis. So, when the given line is rotated 45 degrees clockwise about the x–intercept, the image is the x–axis and its equation is $y = 0$ or $y = 0x + 0$. The required sum is then 0.

11. **80** All of the fourth powers of positive integers less than 100 leave a remainder of 1 when divided by 5, except the multiples of 5. There are 19 multiples of 5 in those 99 numbers, leaving 80 to satisfy the condition of the problem.

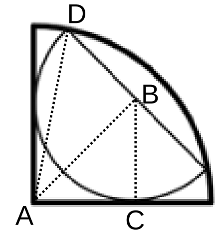
12. **17** Either the number on the green die will be greater than the number on the red die, or vice versa, or the numbers on the dice will be equal. The probability that the numbers are equal is $\frac{1}{6}$. The other two probabilities must be equal because of symmetry. So, the probability that the number on the green die is greater than the number on the red die is $\frac{1-\frac{1}{6}}{2} = \frac{5}{12}$ and the required sum is 17.

13. **60** As the vertical circle rotates, it traces a circular path in the horizontal plane as shown. When it reaches the point A, its x-coordinate is 3 and it will look like an ellipse with a minor axis of 6. We can form a right triangle with a hypotenuse of 6 and a short leg of 3 on the horizontal plane, so the angle of rotation is $60°$ and $n = 60$.



14. **11** Re-write $x + \frac{9}{x-5}$ as $x - 5 + \frac{9}{x-5} + 5$. Then, since $\left((x-5)-3\right)^2 \geq 0 \rightarrow (x-5)^2 + 9 \geq 6(x-5) \rightarrow x - 5 + \frac{9}{x-5} \geq 6 \rightarrow x - 5 + \frac{9}{x-5} + 5 \geq 6 + 5 = 11$.

15. **36** Let point B be the center of the semicircle and let point A be the center of the quarter circle. Let point C be a point of tangency as indicated and point D be a point of intersection of the semicircle and quarter circle. Note that $\overline{AD}$ is a radius of the quarter circle and $\overline{BC}$ is a radius of the semicircle. Since ΔABC is an isosceles right triangle, $AC = 1$ and $AB = \sqrt{2}$. In ΔABD, $BD = 1$ and $AD = \sqrt{3}$. Therefore, the ratio of the area of the quarter circle to that of the semicircle is 3:2. The required product is 36.



Grade Level 11 – NMT 2022 **Solutions**

1. **50** $\frac{2x^{2021}-99x^{2020}}{x^{2019}} = \frac{x^{2019}(2x^2-99x)}{x^{2019}} = 50 \to 2x^2 - 99x = 50 \to 2x^2 - 99x - 50 = 0 \to$ $(2x+1)(x-50) = 0 \to x = -1/2$ or $x = 50$. The positive solution is 50.

2. **6** $f(x) = g(x) \to 10\cos\left(\frac{x}{2}\right) = 10 \to \cos\left(\frac{x}{2}\right) = 1 \to \frac{x}{2} = 2\pi n \to x = 4\pi n$ where n is an integer. In the given interval, $x = -4\pi, 0, 4\pi, 8\pi, 12\pi, 16\pi$. There are 6 points of intersection.

3. **81** In the worst-case scenario, the first 80 selections are only blue or black pens. On the 81st selection there would only be red pens left, thus guaranteeing Kevin has selected a red pen.

4. **3** $z = \left(\frac{3+bi}{2+2i}\right)\left(\frac{2-2i}{2-2i}\right) = \frac{6-6i+2bi+2b}{4+4} = \frac{6+2b+(-6+2b)i}{8}$. In order for z to be a real number, the imaginary part must be equal to 0. Therefore $-6 + 2b = 0 \to b = 3$.

5. **16** $y = 64f(4x-2) = 64f\left(4\left(x - \frac{1}{2}\right)\right)$. The horizontal scale factor is $1/4$ and the vertical scale factor is 64. The required product is $64\left(1/4\right) = 16$.

6. **10** The pyramid has 5 vertices, and any 3 of them may be selected to form an isosceles triangle. Hence there are $\binom{5}{3} = 10$ isosceles triangles that may be formed.

7. **7** $|f(x)| = |14 - 2x|$ is a function with a vertex at $x = 7$. In order for this to be an even function, the vertex needs to be on the $y-$ axis so that $f(x) = f(-x)$. We can accomplish this by shifting the graph 7 units to the left. Therefore $h = 7$.

8. **64** To find the common ratio of the sequence, take the ratio of two consecutive terms: $\frac{4^{x-3}}{4^x} = 4^{-3} = \frac{1}{4^3} = \frac{1}{64}$. The reciprocal of the common ratio is 64.

9. **15** The area of the equilateral triangle is $\frac{1}{2} \cdot 6 \cdot 6 \cdot \sin 60° = 9\sqrt{3}$. The area of each sector of the circle is $\frac{1}{6} \cdot \pi \cdot 3^2 = \frac{3\pi}{2}$. The sum of the areas of the sectors is 3π and the area of the required region is $9\sqrt{3} - 3\pi$. The required sum is $9 + 3 + 3 = 15$.

10. **23** In order for a quadratic function to have no x–intercepts, the discriminant must be less than 0. Therefore $(-b)^2 - 4(3)(12) < 0 \to b^2 - 144 < 0 \to (b-12)(b+12) < 0$. Thus, $-12 < b < 12$ and b must be integral, so there are 23 solutions.

11. **720** Consider "SS" as a group that can be in any of six places. The other letters may be arranged in 5! ways. Thus, the required number of arrangements is $6 \cdot 5! = 720$.

12. **5** There are two conditions for the point(s) whose coordinates are (a, b):
$\left(\sqrt{(a-1)^2+b^2}=2\sqrt{3} \text{ and } a^2-b^2=1\right) \rightarrow (a^2-2a+1+b^2=12 \text{ and } a^2-b^2=1) \rightarrow$
$2a^2-2a+1=13 \rightarrow 2a^2-2a-12=0 \rightarrow a^2-a-6=0 \rightarrow (a-3)(a+2)=0 \rightarrow a=$
3 or $a=-2$.

The required difference is 5.

13. **256** $(\log_{16}x)^2=\log_4 x \rightarrow (\log_{16}x)^2-\log_4(x)=0 \rightarrow \left(\frac{\log_4(x)}{\log_4 16}\right)^2-\log_4(x)=0 \rightarrow \frac{(\log_4(x))^2}{4}-$
$\log_4(x)=0 \rightarrow (\log_4(x))^2-4\log_4(x)=0 \rightarrow (\log_4(x))(\log_4(x)-4)=0 \rightarrow \log_4 x=$
0 or $\log_4 x=4 \rightarrow x=4^0=1$ or $x=4^4=256$.

The required answer is 256.

14. **32** Extend segments $\overline{AD}$ and $\overline{BF}$ and define their intersection as point J. Let $DF=x$, so $CF=3x$ and $AB=4x$. Also let $BE=CE=y$, so $AD=2y$ and $DJ=z$. Since $\Delta BCF \sim \Delta JAB$, $\frac{CF}{AB}=\frac{BC}{AJ}$
$\Rightarrow \frac{3x}{4x}=\frac{2y}{2y+z} \Rightarrow 6y+3z=8y \Rightarrow z=\frac{2}{3}y$. Since $\Delta BGE \sim \Delta JGA$, $\frac{EG}{AG}=\frac{BE}{AJ} \Rightarrow \frac{12}{AG}=\frac{y}{2y+z}$
$\Rightarrow \frac{12}{AG}=\frac{y}{\frac{8}{3}y} \Rightarrow AG=12\left(\frac{8}{3}\right)=32$.

15. **540** $\cos^4 A-\sin^4 A=\frac{\sqrt{2}}{2} \rightarrow (\cos^2 A-\sin^2 A)(\cos^2 A+\sin^2 A)=\frac{\sqrt{2}}{2} \rightarrow \cos^2 A-\sin^2=\frac{\sqrt{2}}{2} \rightarrow$
$\cos(2A)=\frac{\sqrt{2}}{2}$. Therefore, in degrees, either $2A=45+360n$ or $2A=315+360n$, where n is any integer. So, $A=22.5+180n$ or $A=157.5+180n$. In the interval $180 \leq A \leq 360$, $A=202.5$ or $A=337.5$. The required sum is $202.5+337.5=540$.

Grade Level 12 – NMT 2022 Solutions

1. **19** Since $f(x)$ is a real-valued function, $100 - x^2 \geq 0$ and $x^2 - 4 \neq 0$. The first condition yields $-10 \leq x \leq 10$ or 21 integers, and the second condition eliminates 2 of them. Thus, there are 19 integers that satisfy the conditions of the problem.

2. **12** Method 1: Use L'Hôpital's rule: $\lim_{x\to 8} \frac{x-8}{x^{1/3}-2} = \lim_{x\to 8} \frac{1}{\frac{1}{3}x^{-2/3}} = 3(8)^{2/3} = 3(4) = 12$.

   Method 2: Factor the numerator: $\lim_{x\to 8} \frac{(x^{1/3}-2)(x^{2/3}+2x^{1/3}+4)}{x^{1/3}-2} = (8)^{2/3} + 2(8)^{1/3} + 4 = 4 + 4 + 4 = 12$.

3. **72** Let $m\measuredangle ABD = m\measuredangle DBC = x$. Since ΔABC is isosceles, $m\measuredangle C = 2x$. Since $\Delta ABC \sim \Delta BCD$, ΔBCD is also isosceles, so $m\measuredangle BDC = 2x$ as well. Using the sum of the angles of ΔBCD, $x + 2x + 2x = 180° \rightarrow x = 36° \rightarrow m\measuredangle C = 2x = 72°$.

4. **20** Consider factoring $x^4 - 10x^2 + c = (x^2 - p)(x^2 - q)$. This factoring requires $p + q = 10$. Brief trial and error yields $(x^2 - 9)(x^2 - 1) = (x-3)(x+3)(x-1)(x+1) = 0 \rightarrow x = -3, -1, 1, 3$ and the required sum of the squares is 20.

5. **72** Let v be the car's <u>speed</u> for the <u>second</u> trip and let t be the car's <u>time</u> for the <u>first</u> trip. Since the track is the same length for both trips,
   $(v-27)t = v(t-3) \rightarrow vt - 27t = vt - 3v \rightarrow -27t = -3v \rightarrow v = 9t$. As the track is 360 meters long,
   $9t(t-3) = 360 \rightarrow t(t-3) = 40 \rightarrow t = 8 \rightarrow v = 9t = 72$.

6. **2** Method 1: $f'(x) = 4\cos^3 x(-\sin x) - 4\sin^3 x(\cos x) \rightarrow$

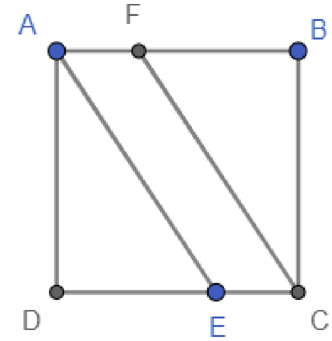
   $$f'\left(\frac{3\pi}{4}\right) = 4\left(-\frac{\sqrt{2}}{2}\right)^3\left(-\frac{\sqrt{2}}{2}\right) - 4\left(\frac{\sqrt{2}}{2}\right)^3\left(-\frac{\sqrt{2}}{2}\right) = 8\left(\frac{\sqrt{2}}{2}\right)^4 = 8\left(\frac{4}{16}\right) = 2.$$

   Method 2: $f(x) = (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) = (1)(\cos 2x) \rightarrow$

   $$f'(x) = -2\sin 2x \rightarrow f'\left(\frac{3\pi}{4}\right) = -2\sin\frac{3\pi}{2} = -2(-1) = 2.$$

7. **620** Let $x = 7$. Then, $2f(7) + f(-6) = 49$. Let $x = -6$. Then $2f(-6) + f(7) = 36$. Solve these equations simultaneously. The result is $f(-6) = \frac{23}{3}$, $f(7) = \frac{62}{3}$, and $30 \cdot f(7) = 620$.

8. **56** If the three regions have equal area, then ΔADE has one-third the area of square *ABCD*. So, $\frac{1}{2}\cdot 84\cdot DE=\frac{1}{3}\cdot 84^2 \rightarrow \frac{1}{2}\cdot DE=\frac{1}{3}\cdot 84 \rightarrow$ $DE=\frac{2}{3}\cdot 84=56$.



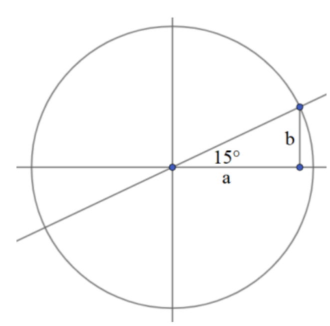
9. **28** Let the dimensions of the rectangle be a and b. Since the diagonal of the rectangle is the diameter of the circle,$a^2+b^2=12^2=144$. Since$(a+b)^2=a^2+b^2+2ab$, $(a+b)^2=144+2\cdot 26=196$. So, $a+b=14$ and the perimeter of the rectangle is 28.

10. **511** If z^2 is 511 units to the left of z, then $z^2=z-511$. Applying the quadratic formula yields $z=\frac{1\pm\sqrt{1-4\cdot 1\cdot 511}}{2}=\frac{1}{2}\pm\frac{\sqrt{4\cdot 511-1}}{2}i$. . So, $a^2+b^2=\frac{1}{4}+\frac{4\cdot 511-1}{4}=\frac{4\cdot 511}{4}$=511.

11. **270** The perimeter $P=6x$ and, since the regular hexagon is made up of 6 equilateral triangles, $A=6\left(x^2\frac{\sqrt{3}}{4}\right)=\frac{3\sqrt{3}}{2}x^2$. Taking d/dt of both sides, $\frac{dA}{dt}=3\sqrt{3}x\frac{dx}{dt}\rightarrow \frac{dA/dt}{dx/dt}=3\sqrt{3}x=\frac{\sqrt{3}}{2}\cdot 6x=\frac{\sqrt{3}}{2}P$.
So, $k=\frac{\sqrt{3}}{2}$ and $360k^2=360\left(\frac{3}{4}\right)=270$.

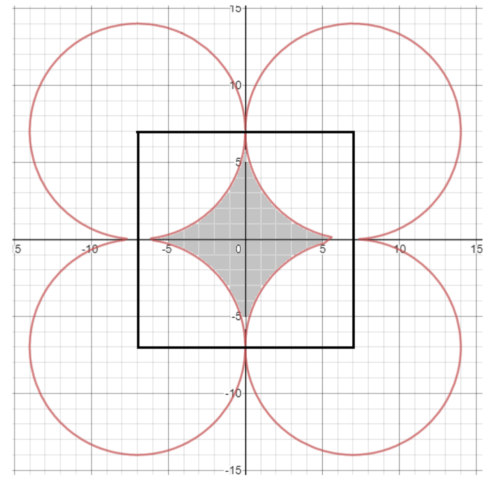
12. **280** The number of permutations of the 8 letters of MATHTEAM (disregarding the space between words) is given by $\frac{8!}{2!2!2!}=7!=5040$. For each 8-letter permutation, there are 7 places you can insert a space to create a unique two-word phrase. Therefore, $N=7\cdot 7!=35280$and $N/1000$ yields a remainder of 280.

13. **314** Let $r=\sqrt{2512}$ be the radius of the circle. A line of the form $y=(\tan\theta)x$ forms an angle of θ with the positive *x*-axis, as shown at right. So, the coordinates (a,b) are $(r\cos 15°, r\sin 15°)$ and the area of the triangle is



$$\frac{1}{2}(r\cos 15°)(r\sin 15°)=\frac{1}{4}r^2\cdot 2\sin 15°\cos 15°=\frac{1}{4}r^2\sin 30°=\frac{1}{8}r^2=\frac{1}{8}(2512)=314.$$

14. **42** Note that the equation is equivalent to $|x|^2+|y|^2-14|x|-14|y|+49=0$. The effect of replacing x with $|x|$ and y with $|y|$ in the equation $x^2+y^2-14x-14y+49=0$ is to take the portion of the graph which lies in quadrant I and reflect it over both the x- and y-axes to create mirror images in all four quadrants. Completing the square gives us $(x-7)^2+(y-7)^2=49$, which is the circle in the upper-right-hand corner of the diagram at right. Reflecting it as shown produces the graph of $x^2+y^2-14|x|-14|y|+49=0$. The shaded region in the middle of the four circles is the desired region, consisting of a square region minus four quarter-circle regions. The area is



$$14^2-49\pi=49(4-\pi)\approx 49\left(4-\frac{22}{7}\right)=49\left(\frac{6}{7}\right)=42.$$

15. **45** Method 1: Since $f'(x)=2x$, L has equation $y=2a(x-a)+a^2+2021$. If L intersects $y=-f(x)$, then $2a(x-a)+a^2+2021=-x^2-2021$ has at least one real solution for x. Expressing as a quadratic in x:

$$2ax-2a^2+a^2+2021+x^2+2021=0$$

$$x^2+(2a)x+\left(4042-a^2\right)=0$$

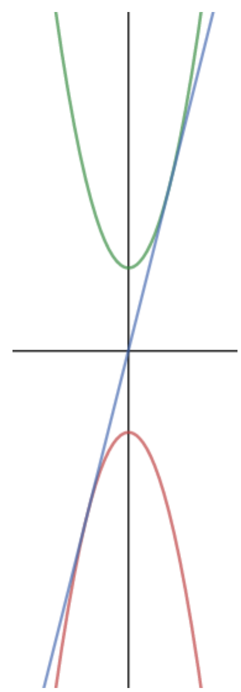
For this to have a real solution for x, the discriminant must be nonnegative:

$$(2a)^2-4(1)\left(4042-a^2\right)\geq 0\rightarrow 4a^2-4\cdot 4042+4a^2\geq 0\rightarrow 8a^2-8\cdot 2021\geq 0\rightarrow a^2\geq 2021.$$

Thus, the minimum value of a is $\sqrt{2021}$. Since $2025=45^2$, a is closest to 45.

Method 2: Graph $y=f(x)$ and $y=-f(x)$, as shown at right (not to scale). If a is too small, the graph of L will go above the graph of $y=-f(x)$. When a is increased to the point at which L intersects the lower curve, L will intersect it at only 1 point, meaning that at the minimum value of a, L will be tangent to both curves. Since the two curves, taken together, have 180° rotational symmetry about the origin, so will their common tangent, and will therefore pass through the origin. So, we need a line L which is tangent to $y=f(x)$ and passes through the origin. As in Method I, L has equation $y=2a(x-a)+a^2+2021$. Plug in $(0,0)$ and solve:

$0=2a(-a)+a^2+2021\rightarrow 0=2021-a^2\rightarrow a=\sqrt{2021}$. Continue as in Method I.



1. **338** The set S contains 26 odd integers and its sum is $1 + 3 + 5 + \ldots + 51 = 26^2 = 676$. The required number is $676/2 = 338$.

2. **96** $f(10) = 70 + a$ and $g(f(10)) = \sqrt{70 + a - 22} = \sqrt{a + 48} = 12 \rightarrow a + 48 = 144 \rightarrow a = 96$.

3. **72** David works at a rate of ½ of a driveway per hour while William works at a rate of ⅓ driveway/hour. Together they can shovel ⅚ driveway/hour. To shovel one driveway together takes 1/⅚ hours or 72 minutes.

4. **242** The diagonals of a rhombus are perpendicular bisectors of each other and form four congruent right triangles whose leg lengths are 9 and 40. Since these four triangles are congruent, their hypotenuses (the sides of the rhombus) have the same length of 41. Each triangle has an altitude to the hypotenuse that must also be a radius of the inscribed circle. The area of each of the right triangles is $\frac{1}{2} \cdot 9 \cdot 40 = 180$. Since $\frac{1}{2} \cdot r \cdot 41 = 180, r = \frac{360}{41}$. The area of the inscribed circle is $\pi r^2 \approx 242$.

5. **40** Let $a_2 = x$, then $a_3 = \frac{2+x}{2} = \frac{x}{2} + 1, a_4 = \frac{1}{3}\left(x + 2 + \frac{x+2}{2}\right) = \frac{x}{2} + 1, a_5 = \frac{1}{4}\left(x + 2 + \frac{x+2}{2} + \frac{x}{2} + 1\right) = \frac{x}{2} + 1, \ldots$. We see that all subsequent terms will be equal, so $a_{10} = \frac{x}{2} + 1 = 21$ and $x = 40$.

6. **446** $(x + y)^3 = x^3 + y^3 + 3x^2y + 3xy^2 = 10 + 3xy(x + y) = 10 + 36xy$

$$12^3 = 1728 = 10 + 36xy \rightarrow xy = \frac{1718}{36} = \frac{859}{18}$$

$$(x + y)^2 = x^2 + y^2 + 2xy \rightarrow 12^2 = x^2 + y^2 + 2\left(\frac{859}{18}\right) \rightarrow x^2 + y^2 = 144 - \frac{859}{9} = \frac{437}{9}$$

and the required sum is 446.

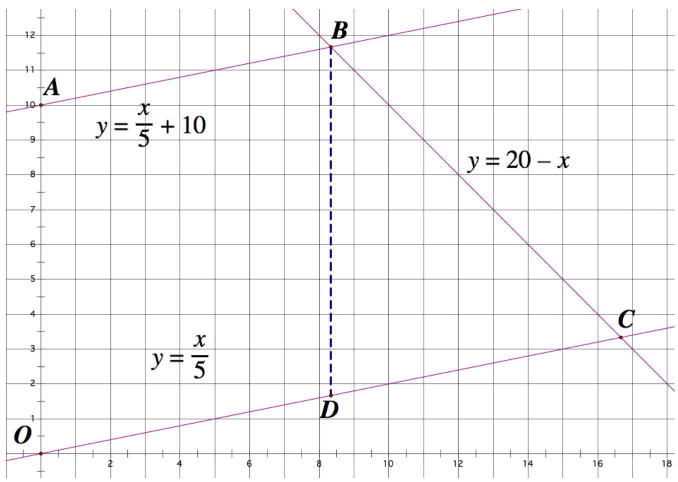
7. **91** Call the roots of $x^3 - ax^2 + 2111x - 2021 = 0$, p, q, and r. So, $x^3 - ax^2 + 2111x - 2021 = (x - p)(x - q)(x - r)$. Since the only prime factors of 2021 are 43 and 47, the three roots must be 43, 47, and 1. Their sum is 91 and that is the value of the coefficient a.

8. **98** The probability that Davesh wins any round is ⅓. The required probability is the same as the probability that Davesh wins either 3, 4 or 5 of the rounds. This is a Bernoulli experiment. P(Davesh wins exactly 3 rounds out of 5) $= \binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \frac{40}{243}$ because Davesh must win three rounds and lose the other two and these can happen in any order. Similarly,
P(Davesh wins exactly 4 rounds out of 5) $= \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1 = \frac{10}{243}$
P(Davesh wins all 5 rounds) $= \left(\frac{1}{3}\right)^5 = \frac{1}{243}$

Since any of the above possibilities can occur, we need to sum the probabilities: $\frac{51}{243}$. This reduces to $\frac{17}{81}$. The required sum is 98.

9. **420** Note that $r = \frac{1}{20+\frac{1}{21+r}} \longrightarrow \frac{1}{r} = 20 + \frac{1}{21+r} \longrightarrow r + 21 = 20\,r\,(r + 21) + r \longrightarrow 20r^2 + 420r - 21 = 0 \longrightarrow \frac{-210 \pm \sqrt{210^2+4 \cdot 21 \cdot 20}}{20}$. Since r is given to be positive, $r = \frac{-210+\sqrt{210^2+4 \cdot 21 \cdot 20}}{20}$ takes the form $m + p\sqrt{q}$ and both b and c are integers, then the other root is $\frac{-210-\sqrt{210^2+4 \cdot 21 \cdot 20}}{20}$. The sum of the roots $= -\frac{b}{a} = \frac{-420}{20} \longrightarrow b = 420$.

10. **125** Using algebra or the calculator, the coordinates of point B are (25/3, 35/3) and the coordinates of point C are (50/3, 10/3). Separate the trapezoid into a parallelogram and a triangle as follows: Through point B, draw a line parallel to the y- axis. The line intersects $\overline{OC}$ at point D whose coordinates are (25/3, 5/3). Then the area of the parallelogram $OABD$ is $10 \cdot \frac{25}{3} = 250/3$. The area of $\Delta BDC = \frac{1}{2} \cdot BD \cdot h_{\overline{BD}} = \frac{1}{2} \cdot 10 \cdot \frac{25}{3} = 125/3$. The area of the trapezoid is the sum of the area of the parallelogram and the area of the triangle. The area is 125.



Team Problem Solving – NMT 2022 Solutions

1. **There are several arrangements that work.** Here is one such arrangement:

2	2	0	1
0	1	2	2
1	0	2	2
2	2	1	0

2. **7** When we examine the gaps between two-digit primes, we find only one gap that contains more than 5 integers. There are 7 composite integers between the primes 89 and 97.

3. **144** Start by letting $\frac{2}{3}=\frac{1}{3}+\frac{1}{3}$ and then look for two unit fractions that equal $\frac{1}{3}$. Since $\frac{1}{4}<\frac{1}{3}$ subtract to get that $\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$. It follows that $\frac{2}{3}=\frac{1}{3}+\frac{1}{4}+\frac{1}{12}$ and $3 \times 4 \times 12 = 144$. Of course, there are other triples (a, b, c) that work; e.g., (2,7,42) or (2,8,24). But, their sum exceeds ours.

4. **15** Arrange the 7 ones in a row. There are 6 spaces between the seven 1's. Select 4 of those spaces to fill with 0's. Therefore, there are ${}_6C_4=\frac{6!}{2!\cdot 4!}=15$ numbers that can be formed as required.

5. **59** Let A, B, and C be the number of each type of puzzle. Then $A+B+C=100$ and in terms of cents, $40A+70B+100C=8770$. Multiply the first equation by 70 and subtract the result from the second equation. The result is $-30A+30C=1770$. Therefore $C-A=59$.

6. **682** The place values of a binary system are all powers of 2, with the right-most digit being $2^0 = 1$ and the exponents increasing by 1 as the digits form to the left. Therefore, 1010101010_2 equals the sum $2^9+2^7+2^5+2^3+2^1=512+128+32+8+2=682$.

7. **49** A theorem in geometry states that for any point in the interior of a rectangle the sum of the squares of the lengths of segments drawn from a pair of opposite corners to the point will equal the sum of the squares of the lengths of the segments drawn from the remaining two vertices to the same point. Therefore, $(MA)^2+(MC)^2=(MH)^2+(MR)^2$ so $(MA)^2=15^2+20^2-24^2=625-576=49$.
[To prove the theorem draw lines parallel to the sides of the rectangle through the selected interior point and apply the Pythagorean theorem several times.]

8. **673** Since $\log_2(\log_4(\log_5(p)))=\log_4(\log_5(\log_2(q)))=\log_5(\log_2(\log_4(r)))=0$, we know that $\log_4(\log_5(p))=2^0=1 \rightarrow \log_5(p)=4 \rightarrow p=5^4=625$, $\log_5(\log_2(q))=4^0=1 \rightarrow \log_2(q)=5 \rightarrow q=2^5=32$, and $\log_2(\log_4(r))=5^0=1 \rightarrow \log_4(r)=2 \rightarrow r=4^2=16$. Thus $a+b+c=625+32+16=673$.

9. **98** All unit fractions lie in the interval $(0, \frac{1}{2}]$. There are no unit fractions between 2 and $\frac{1}{2}$. Only $\frac{1}{2}$ lies between 3 and $\frac{1}{3}$. Only $\frac{1}{3}$ and $\frac{1}{2}$ lie between 4 and $\frac{1}{4}$. In general, for every whole number n there are $n-2$ unit fractions between n and $\frac{1}{n}$. Hence, when $n=100$, there are 98 unit fractions.

10. **14** Since the square pyramid is inscribed in the cube, its volume will be $\frac{1}{3}$ the volume of the cube or $\frac{56\pi}{3}$ cm^3. The volume of a sphere of radius r is equal to the volume of the pyramid so $V = \frac{4}{3}\pi r^3 = \frac{56\pi}{3}$ and $r^3 = 14$. The volume of the second cube whose edge is equal to r is also r^3

11. **9** The only time an integer power of 2 can equal an integer power of 3 is when both powers are 0. We need to find the common solution to the equations: $x^2 - 8x - 9 = 0$ and $2x^2 - 19x + 9 = 0$. Solving each equation results in: $x^2 - 8x - 9 = 0 \rightarrow (x - 9)(x + 1) = 0 \rightarrow x = 9$ or -1 and $2x^2 - 19x + 9 = 0 \rightarrow (x - 9)(2x - 1) = 0 \rightarrow x = 9$ or $1/2$. The common solution is 9.

12. **4** The area of the entire letter "F" is 32 square units. We need to separate the letter into two regions whose areas will each be 16 square units. The x-axis creates an upper region of area 14 square units. Therefore, we need to angle the line down to create an additional 2 square units. Since that part of the "F" is basically a 2×4 rectangle we need to cut it in quarters with a line whose slope is $-1/4$. The negative reciprocal of $-1/4$ is 4.

13. **243** Convert all of the numbers to base 3 to result in $\sqrt{\frac{3^{24}+3^{16}}{3^6+3^{14}}} = \sqrt{\frac{3^{16}\left(3^8+1\right)}{3^6\left(1+3^8\right)}} = \sqrt{3^{10}} = 3^5 = 243$.

14. **29** Let the three integers be $n - 2$, n, and $n + 2$. Solve the equation $(n - 2)(n)(n + 2) - 24 = n^3 - 140$. The result is $n^3 - 4n - 24 = n^3 - 140 \rightarrow 4n = 116 \rightarrow n = 29$. The mean of the numbers $n - 2$, n, and $n + 2$ is $n = 29$.

15. **42** Since the shortest side of the largest rectangle is 6 its length must be 9. The side measures are all integers, so the width of the smallest rectangle is 2 and the width of the middle-sized rectangle is 4. Their lengths are 3 and 6 respectively. The perimeter of the figure is $6 + 12 + 2 + 3 + 4 + 15 = 42$.

16. **20** The fact that $2\sin x = 5\cos x$ means that $\tan x = 5/2$. In a right triangle with an angle whose measure is x the side opposite the angle can be 5 and the adjacent side 2. Therefore, the hypotenuse is $\sqrt{29}$. Thus, $\sin x = \dfrac{5}{\sqrt{29}}$ and $\cos x = \dfrac{2}{\sqrt{29}}$ so $58 \sin x \cos x = 58 \cdot \frac{5}{\sqrt{29}} \cdot \frac{2}{\sqrt{29}} = 58 \cdot \frac{10}{29} = 20$.

Alternate Solution: Square both sides of the original equation resulting in $4\sin^2 x = 25\cos^2 x$ and then substitute $1 - \cos^2 x$ for $\sin^2 x$. So, $4 - 4\cos^2 x = 25\cos^2 x \rightarrow \cos^2 x = 4/29$. Multiply both sides of the original equation by $\cos x$ to get $2\sin x \cos x = 5\cos^2 x = 5(4/29) = 20/29 \rightarrow$ $58\sin x \cdot \cos x = 29(20/29) = 20$.

17. **19** Since the product is $144 = 2^4 \cdot 3^2$ there are $5 \cdot 3 = 15$ possible factors. Since the sum of the squares of the three numbers is 149, the largest possible number is 12. If we assume 12 is one the integers, the sum of the squares of the other two must be 5, so 1 and 2 would be the numbers. Unfortunately, the product of $1 \times 2 \times 12 = 24$ which is not 144. Try the next largest possible value, 9. Then $149 - 81 = 68$ and the other two numbers would be 8 and 2. The product of these three numbers $9 \times 8 \times 2 = 144$ the desired result. Thus, the sum of the three numbers is $2 + 8 + 9 = 19 = A_{17}$.

18. **4** Since $|x - 2| = A_{17} = 19$, it follows that $x - 2 = 19$ or $x - 2 = -19$. In the first case $x = 21$ and in the second case $x = -17$. Their sum is $21 + (-17) = 4 = A_{18}$. The result is actually independent of the value of A_{17}.

19. **6** Label the third sides of the two triangles c and d. The goal is to maximize $c - d$. So we want c to be as great as possible and d to be as small as possible. In any triangle the sum of the lengths of the two shortest sides must be greater than the length of the third side. Both triangles have sides of length $A_{18} = 4$ and share a side of length of length b. To maximize c, we want c to be the longest side of the triangle which means that c is 1 less than $b + 4$ or $c = b + 3$. To minimize d we want d to be the shortest side of the triangle which means that d is one more than $b - 4$ or $d = b - 3$. The difference $c - d = (b + 3) - (b - 3) = 6$.

20. **45** Since $A_{19} = 6$, the area of the region between the two squares equals $6^2 - 4^2 = 36 - 16 = 20$. The portion outside the given square is composed of four 1×6 rectangles and four quarter sectors of the disc. The sum of the areas of the four rectangles is $4(1 \times 6) = 24$ and the sum of the areas of the four quarter circles is the area of the circle which $= \pi$. Thus, the total area of the region determined by the disc $= 20 + 24 + \pi = 44 + \pi = a + b\pi$. Consequently, $a = 44$ and $b = 1$ so $a + b = 45$.