Grade Level 9 - NMT 2020 **Solutions**

1. **196** $C = \pi d = 28\pi \rightarrow d = 28 \rightarrow r = 14$. So, $A = \pi r^2 = \pi(196)$. Thus, $k = 196$.

2. **45** Let x, $x+2$, and $x+4$ represent the integers. So, $x^2 = 9 + 5(x+2+x+4) \rightarrow$ $x^2 = 9 + 5(2x+6) \rightarrow x^2 - 10x - 39 = 0 \rightarrow (x-13)(x+3) = 0 \rightarrow x = 13$, only. Thus, the required sum is $13 + 15 + 17 = 45$.

3. **1** For line l: $6x - 3y = -2 \rightarrow y = 2x + \frac{2}{3}$. Since $l \parallel m$, each line has a slope of 2. For line m: $ax - by = 5 \rightarrow y = \frac{a}{b}x - \frac{5}{b}$. Since the y-intercept is -10, $-\frac{5}{b} = -10 \rightarrow b = \frac{1}{2}$. So, $\frac{a}{b} = 2 \rightarrow \frac{a}{\frac{1}{2}} = 2 \rightarrow a = 1$.

4. **20** $5\sqrt{147} - 4\sqrt{108} + 2\sqrt{27} = 5\sqrt{49}\sqrt{3} - 4\sqrt{36}\sqrt{3} + 2\sqrt{9}\sqrt{3} = 35\sqrt{3} - 24\sqrt{3} + 6\sqrt{3} =$ $17\sqrt{3}$. Thus, $17 + 3 = 20$.

5. **6** $(2^4)^{0.5x+1.5} = 2^{3x} \rightarrow 2^{2x+6} = 2^{3x} \rightarrow 2x + 6 = 3x \rightarrow x = 6$.

6. **3** Since $\measuredangle B$ is the right angle, $\overline{AC}$ is the hypotenuse. By the Pythagorean Theorem, $(AB)^2 + (BC)^2 = (AC)^2 \rightarrow \left(8\sqrt{2}\right)^2 + (BC)^2 = 12^2 \rightarrow 128 + (BC)^2 = 144 \rightarrow BC = 4$. So, $\sin A = \frac{4}{12} = \frac{1}{3} \rightarrow k = 3$.

7. **10** $2x^2 - 26x + 60 = 0 \rightarrow 2(x^2 - 13x + 30) = 0 \rightarrow 2(x-10)(x-3) = 0 \rightarrow x = 10$ or $x = 3$. Thus, the greater zero is 10.

8. **9** The midpoint of diagonal $\overline{AC}$ is $\left(\frac{6+10}{2}, \frac{9+3}{2}\right) = (8, 6)$, and its slope is $\frac{9-3}{6-10} = -\frac{3}{2}$. The diagonals of a square are perpendicular bisectors of each other, so the slope of diagonal $\overline{BD}$ is $\frac{2}{3}$. From the midpoint $(8, 6)$, follow the slope of $\overline{BD}$ in both directions to get $(11, 8)$ and $(5, 4)$, the square's other two vertices. Thus, the vertex closest to the origin is $(5, 4)$ and $5 + 4 = 9$.

9. **3** If $\frac{x+y}{2} = z$, $\frac{x+z}{2} = y + 1$, and $\frac{y+z}{2} = 2x + 2$, then by the addition property, $\frac{2x+2y+2z}{2} = z + y + 1 + 2x + 2 \rightarrow x + y + z = z + y + 2x + 3 \rightarrow x = 2x + 3 \rightarrow x = -3$. Thus, $|x| = 3$.

10. **5** Using integral factors, the number $18 = 1 \cdot -1 \cdot 2 \cdot 3 \cdot -3$. Thus, $n = 5$.

11. **505** Each successive pair in the numerator can be factored as the difference of two squares:
$$\frac{(100+99)(100-99)+(98+97)(98-97)+\cdots+(4+3)(4-3)+(2+1)(2-1)}{10}.$$
When simplified, the numerator is an arithmetic series whose sum is $\frac{n}{2}(a_1 + a_n)$:
$$\frac{100+99+98+97+\cdots+4+3+2+1}{10} = \frac{\frac{100}{2}(100+1)}{10} = 505.$$

12. **88** By factoring the sum of two cubes, $x^3 + y^3 = (x+y)(x^2 - xy + y^2) = 819 \rightarrow$ $3(x^2 - xy + y^2) = 819 \rightarrow x^2 - xy + y^2 = 273$. Adding $3xy$ to both sides yields $x^2 + 2xy + y^2 = 273 + 3xy \rightarrow (x+y)^2 = 273 + 3xy \rightarrow 9 = 273 + 3xy \rightarrow xy = -88$. Thus, $|xy| = 88$.

13. **48** Since perfect square factors have even exponents, 2^7 has 4 perfect square factors: 2^0, 2^2, 2^4, and 2^6. The number 3^4 has 3 perfect square factors, 5 has 1, 7^2 has 2, and 11^3 has 2. Thus, the number of perfect square factors of the given product is $4 \cdot 3 \cdot 1 \cdot 2 \cdot 2 = 48$.

14. **34** By letting $x = 1$ and $y = 1$ in $20x + 3y = 2003$, the first and the last of the first-quadrant lattice points are $(1,\ 661)$ and $(100,\ 1)$. The slope of the given line is $-\frac{20}{3}$, so from the lattice point $(1,\ 661)$, the y-coordinates will decrease by 20 while the x-coordinates increase by 3 creating the sequence of lattice points: $(1,\ 661)$, $(4,\ 641)$, $(7,\ 621), \ldots, (97,\ 21)$, $(100,\ 1)$. Using the arithmetic sequence formed by the x-coordinates, and the formula $a_n = a_1 + d(n-1)$, $100 = 1 + 3(n-1) \rightarrow 99 = 3(n-1) \rightarrow 33 = n - 1 \rightarrow n = 34$. (Note: Using the y-coordinates yields the same result: $1 = 661 - 20(n-1) \rightarrow n = 34$)

15. **437** There are 15 ways that the four dice throws occur in strictly increasing order: 1234, 1235, 1236, 1245, 1246, 1256, 1345, 1346, 1356, 1456, 2345, 2346, 2356, 2456, and 3456. This result can also be computed using the combination ${}_6C_4 = \frac{6!}{4! \cdot 2!} = 15$. Since each die throw has 6 possible outcomes, there are 6^4 ways the dice can be tossed. So, the required probability is $\frac{15}{6^4} = \frac{15}{1296} = \frac{5}{432}$, and $5 + 432 = 437$.

Grade Level 10 - NMT 2020 **Solutions**

1. **108** The prime factors of 2020 are 2, 5, and 101. The required sum is 108.

2. **16** $\frac{-\frac{4}{3^2}+\frac{1}{3}+3}{1-\frac{1}{3}} = \frac{-4+3+27}{9-3} = \frac{26}{6} = \frac{13}{3}$ and the required sum is 16.

3. **5** Each time Noah drinks half the contents of the container, he drinks half the juice. So, the 20% juice drink first became a 10% juice drink and then became a 5% juice drink. So, $p = 5$. Alternatively, suppose the container holds 1000 ml; initially, 200 ml juice and 800 ml water. After Noah drinks, he leaves 100 ml juice and 400 ml water. He fills the container to 100 juice and 900 ml water, then drinks it to 50 ml juice and 450 ml water, and then fills it to 50 ml juice and 950 ml water. Again, 50/1000 = 5%.

4. **119** If $n = 0.2020202020\ldots$, then $100n = 20.20202020\ldots$. Subtraction yields $99n = 20 \rightarrow n = \frac{20}{99}$. The required sum is 119.

5. **34** The roots of $x^2 - 5x - 6 = 0$ are $r = 6$ and $s = -1$. So, the roots of $x^2 - ax + b = 0$ are 12 and -2. Therefore, the sum of the roots, $a = 10$, and the product of the roots, $b = -24$. Thus, $a - b = 34$.

6. **4** Translate points C and A as follows: $T_{3,2}(-3,-2) = C'(0,0)$ and $T_{3,2}(3,1) = A'(6,3)$. Rotate point A' 90° counterclockwise about the origin, C': $R_{90°,C'}(6,3) = A''(-3,6)$. Finally, by reversing the initial translation, $T_{-3,-2}(-3,6) = (-6,4)$. Thus, $k = 4$.

7. **15** If $x = 4$, $f(2) = 4^2 - 4 \cdot 4 + 15 = 15$

8. **11** Triangles ADE, DEF, and ABC are 30-60-90 triangles. Since $DE = \sqrt{3}$, and $DEFB$ is a rectangle, $AD = 1$, $EF = DB = 3$, and $AB = 4$. So, $BC = 4\sqrt{3}$ and the area of $\Delta ABC = \frac{1}{2}(4)(4\sqrt{3}) = 8\sqrt{3}$. Thus, $8 + 3 = 11$.

9. **13** The product of the roots is 130 and 130 is the product of the primes 2, 5, and 13. The greatest of these is 13.

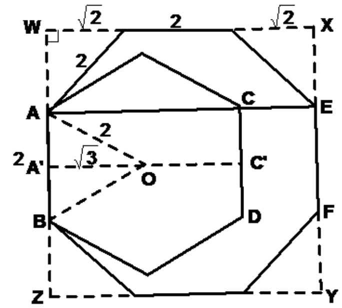
10. **0** The segment passing through the four points of tangency, $\overline{AEGC}$, is a diameter of the large circle and its length is 6. So, the length of semicircle ABC is $b = 3\pi$. Arcs ADE, EFG, and GHC are semicircles of the small circles, so their lengths are each π. Thus, $s = 3\pi$ and $b - s = 3\pi - 3\pi = 0$.

11. **889** The required probability is ${}_6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 = \frac{6!}{3!\cdot 3!}\left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 = \frac{160}{729}$. The required sum is 889.

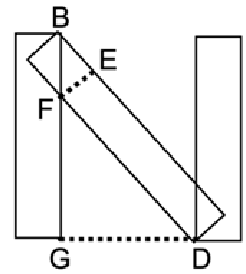
12. **11** Note that $45_7 = 4 \cdot 7^1 + 5 \cdot 7^0 = 33$ and $302_5 = 3 \cdot 5^2 + 0 \cdot 5^1 + 2 \cdot 5^0 = 77$. The greatest common factor of 33 and 77 is 11.

13. **2** The vertex of the parabola $y = a(x+b)^2 + c$ is $(-b, c)$ and it lies on the line $y = -2$. So, $c = -2$. The lines $y = -2x - 1$ and $y = 2x - 5$ are symmetric to the line $x = 1$, the parabola's axis of symmetry, so the vertex is $(1,-2)$ and $b = -1$. Since $(0,-1)$ lies on the parabola, $-1 = a(0-1)^2 - 2 \rightarrow a = 1$. Thus, $abc = (1)(-1)(2) = 2$.

14. **22** Point O is the center of the regular hexagon whose side lengths are 2. Triangle AOB is an equilateral triangle with altitude $OA' = \sqrt{3}$. So, $A'C' = AC = 2\sqrt{3}$. Circumscribe square $WXYZ$ about the regular octagon whose side lengths are also 2. Each triangle in the square's corners is an isosceles right triangle with hypotenuse 2 and leg $\sqrt{2}$. So, $WX = AE = 2 + 2\sqrt{2}$. Thus, the distance between $\overline{CD}$ and $\overline{EF}$, $CE = AE - AC = 2 + 2\sqrt{2} - 2\sqrt{3} = 2 + \sqrt{8} - \sqrt{12}$. Therefore, $2 + 8 + 12 = 22$.



15. **115** Construct $\overline{FE}$ perpendicular to $\overline{BE}$. We know that $FE = 1$, $GD = 3$, and it can be proved that $\angle FBE \cong \angle DFG$. Let $BF = x$. Then, $FG = 5 - x$ and by similar triangles, $FD = 3x$. By the Pythagorean Theorem, $3^2 + (5-x)^2 = (3x)^2 \rightarrow x = \frac{3\sqrt{33}-5}{8}$. So, the required area is $5 + 3\left(\frac{3\sqrt{33}-5}{8}\right) + 5 = \frac{65+9\sqrt{33}}{8}$ and $a + b + c + d = 115$



Grade Level 11 - NMT 2020 **Solutions**

1. **22** Since $f(x)$ is odd, $g(9) = f(5) = -f(-5) = -(-22) = 22$.

2. **31** $(7+5i)(3-2i) + \frac{2-2i}{2+2i} = 31 + i + \frac{-8i}{8} = 31$.

3. **110** All divisors of 14^{10} have the form $2^a \cdot 7^b$ with integers $0 \le a \le 10$ and $0 \le b \le 10$. Because the divisors are required to be even, they need further restriction: $1 \le a \le 10$ and $0 \le b \le 10$. Thus 14^{10} has $10 \cdot 11 = 110$ divisors.

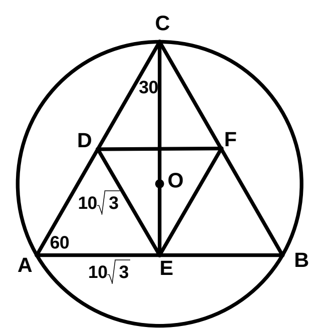
4. **27** $\sqrt{2x-4} - \sqrt{x+5} = \sqrt{x-25} \to 2x - 4 + x + 5 - 2\sqrt{(2x-4)(x+5)} = x - 25 \to 2x + 26 = 2\sqrt{(2x-4)(x+5)} \to x + 13 = \sqrt{(2x-4)(x+5)} \to x^2 + 26x + 169 = 2x^2 + 6x - 20 \to 0 = x^2 - 20x - 189 \to 0 = (x-27)(x+7) \to x = 27$. Since the domain of the given equation is $\{x : x \ge 25\}$, the solution checks.

5. **10** $3\cos^2 x - 8\cos x + 4 = 0 \to (3\cos x - 2)(\cos x - 2) = 0 \to \cos x = \frac{2}{3}$, $\cos x \ne 2$. Since x is in quadrant I, $\sin x = \sqrt{1 - \left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$. So, $\tan x = \frac{\sin x}{\cos x} = \frac{\sqrt{5}}{2}$. The required product is 10.

6. **13** The question requires each of the following to be true: $2^7 \cdot 2^{15} > 2^k$, $2^7 \cdot 2^k > 2^{15}$, and $2^{15} \cdot 2^k > 2^7$ $\to 7 + 15 > k$ and $7 + k > 15$ and $15 + k > 7$ $\to k < 22$ and $k > 8$ and $k > -8$. Therefore, k must be an integer between 8 and 22. There are 13 such values of k.

7. **5** Working in miles per hour, let t be the time it takes in hours for her to get to school on time. Represent the distance she travels to school by bike as $10\left(t + \frac{1}{6}\right)$ miles and the distance she travels by car as $25\left(t - \frac{2}{15}\right)$ miles. Set these equal and solve to get $t = \frac{1}{3}$. Therefore, $d = 10\left(\frac{1}{3} + \frac{1}{6}\right) = 5$ miles.

8. **400** Since ΔADE is equilateral, $AE = 10\sqrt{3}$. Since ΔACE is 30-60-90, $CE = 30$. Since point O is the intersection of the medians of ΔABC, $\overline{CO}$ is the radius of circle O and its length is two-thirds of CE and equal to 20. Thus, the area of the circle is 400π, and $k = 400$.



9. **6** $(x^2 + 3x + 2)(x^2 - x - 6) \le 0 \to (x+2)(x+1)(x-3)(x+2) \le 0$. Checking intervals on a number line, the solution set is $-2 \cup [-1,3]$. This set contains 6 integral solutions; namely $-2, -1, 0, 1, 2$, and 3.

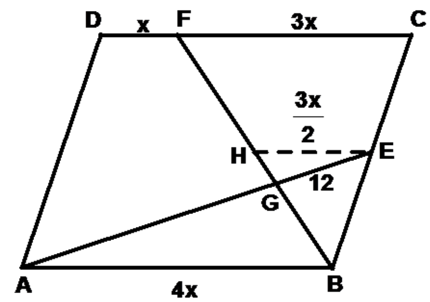
10. **62** Using a law of logarithms, $3\log_8 x = 7 - \frac{2}{\log_8 x} \to 3(\log_8 x)^2 - 7(\log_8 x) + 2 = 0 \to (3\log_8 x - 1)(\log_8 x - 2) = 0 \to \log_8 x = \frac{1}{3}$ or $2 \to x = 2$ or 64. The required difference is 62.

11. **12** $(n+2)! - 2n! = 180n! \to (n+2)! = 182n! \to (n+2)(n+1) = 182 \to n^2 + 3n - 180 = 0 \to (n+15)(n-12) = 0 \to n = 12$.

12. **7** By the definition of a geometric sequence, $\frac{\sin x}{1} = \frac{\sin x \cos 2x}{\sin x}$. Since none of the terms of the sequence is 0, $\sin x \neq 0$, so $\sin x = \cos 2x \rightarrow \sin x = 1 - 2\sin^2 x \rightarrow 2\sin^2 x + \sin x - 1 = 0 \rightarrow$ $(2\sin x - 1)(\sin x + 1) = 0 \rightarrow \sin x = \frac{1}{2}$ or $-1 \rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$. The sum of these roots is $\frac{5\pi}{2}$, so $5 + 2 = 7$.

13. **105** Let x be the number of students in the school that like math and let y be the number of students in the school that do not like math. The resulting equation is $x + y = 200$. Let $0.2y$ be the number of students in the seventh grade who do not like math and let $0.3x$ be the number of students in the eighth grade who like math. Thus $0.7x$ represents the number of students in the seventh grade who like math. The resulting equation is $0.7x + 0.2y = 0.575 \cdot 200$. If the two equations are solved simultaneously, the result is $(150, 50)$. The number of seventh grade students who like math is $0.7 \cdot 150 = 105$.

14. **32** From point E, draw $\overline{EH} \parallel \overline{CF}$ so that point H is on $\overline{BF}$. Since point E is the midpoint of $\overline{BC}$, then H is the midpoint of $\overline{BF}$ and $EH = \frac{1}{2}CF$. If $CF : FD = 3$, then $CF = 3x$, $FD = x$, $EH = \frac{3x}{2}$, and $AB = 4x$. Using alternate interior angles, $\Delta ABG \sim \Delta EHG$, so $\frac{AG}{12} = \frac{4x}{\frac{3x}{2}} \rightarrow \frac{AG}{12} = \frac{8}{3} \rightarrow AG = 32$.



15. **864** The number of ways of creating 3 true-false questions without all having the same answer is 6. The number of ways of creating 4 multiple choice questions in any order with the 3 choices in any order is $4! \cdot 3! = 144$. Thus, $144 \cdot 6 = 864$.

Grade Level 12 - NMT 2020 **Solutions**

1.
2.
3.
4.
5.
6.
7.
8.
9.
10.
11.
12.
13.
14.
15.

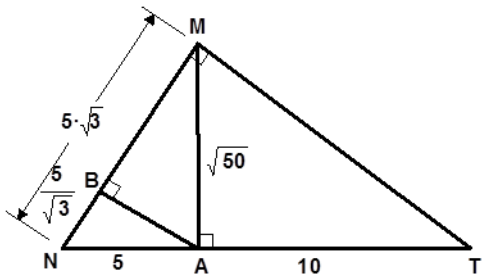
1. **508** Divide 2020 by 4 to find that the average of these four integers is 505. Thus, the four consecutive even integers are 502, 504, 506, and 508, the greatest of which is 508. Alternatively, if $x-2$, x, $x+2$, and $x+4$ represent the integers, $4x+4=2020 \rightarrow x=504$. Thus, $x+4=508$.

2. **19** The left side of the equation can be viewed as the difference of two squares. When it is factored, the result is $[(7x-5)+(10x+3)][(7x-5)-(10x+3)] = (17x-2)(-3x-8)=0 \rightarrow x=\frac{2}{17}$ or $x=-\frac{8}{3}$. The required sum is $2+17=19$.

3. **184** If $AP : PB = 3 : 5$, then $AP=\frac{3}{8}(AB)$. So, $50-20=\frac{3}{8}(x-20)$ and $44-20=\frac{3}{8}(y-20)$. Solving each equation yields $x=100$ and $y=84$. Thus, $x+y=184$.

4. **505** Note that $2020n = 2^2 \cdot 5 \cdot 101n$. In order to make this expression into the least possible perfect square, n must contain exactly one factor of 5 and exactly one factor of 101. The required answer is 505.

5. **408** Dividing by 100, let $NA=5$ and $AT=10$ to simplify computation. Since the altitude drawn to the hypotenuse of right ΔNMT is the mean proportional between the segments of the hypotenuse, $\frac{5}{MA}=\frac{MA}{10} \rightarrow MA=\sqrt{50}$. By the Pythagorean Theorem in ΔNAM, $5^2+\sqrt{50}^2=(NM)^2 \rightarrow NM=5\sqrt{3}$. In right ΔNAM, each leg is the mean proportional between the hypotenuse and the projection of that leg on the hypotenuse. So, $\frac{5\sqrt{3}}{5}=\frac{5}{NB} \rightarrow NB=\frac{5}{\sqrt{3}}$. Thus, by the Pythagorean Theorem in ΔNAB, $(AB)^2+\left(\frac{5}{\sqrt{3}}\right)^2=5^2 \rightarrow AB=\sqrt{\frac{50}{3}} \approx 4.082$. Finally, $100(AB)=408$, rounded to the nearest integer.



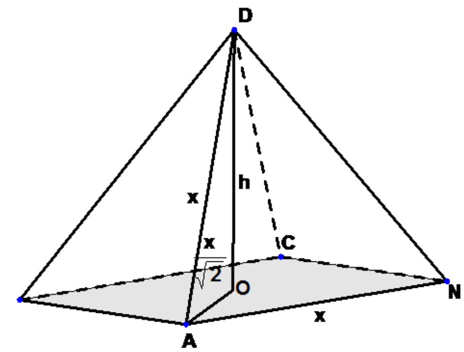
6. **527** Another equation for circle N can be obtained by completing the square: $(x+6)^2+(y+10)^2=16$. Thus, the center of circle N is the point whose coordinates are $(-6,-10)$. Its radius is 4. The center of circle M is the point whose coordinates are $(12,4)$ and its radius is 3. Draw the line of centers, $\overline{MN}$. The required minimum distance between the circles will be the positive difference between MN and the sum of the radii. Calculate $MN=\sqrt{520}$ using either the Pythagorean Theorem or the distance formula. The minimum distance between the circles is $\sqrt{520}-7$ and the required sum is 527.

7. **480** The equations of lines n and m are, respectively, $y-20=4(x-20)$ and $y-20=\frac{1}{4}(x-20)$. Intersecting each of these lines with the given line, $x+y=80$, yields the points $(28,\ 52)$ and $(52,\ 28)$, the endpoints of the triangle's base. Since these points are symmetric about the line $y=x$, the given triangle is isosceles with an altitude on the line $y=x$ that passes through vertex $(20,20)$ and the midpoint of the triangle's base, $(40,40)$. So, the area of the triangle is $\frac{1}{2} \cdot \sqrt{(28-52)^2+(52-28)^2} \cdot \sqrt{(40-20)^2+(40-20)^2} = \frac{1}{2}\left(24\sqrt{2}\right)\left(20\sqrt{2}\right)=480$.

8. **21** There are $2^5 = 32$ distinct game outcomes. This is a complete list of outcomes where Cindy wins with a majority of red cards: RRRRR, BRRRR, RRRRB, RRRBB, BBRRR. Similarly, there are 5 additional mirrored outcomes in which Cindy wins with a majority of black cards. Therefore, the probability that Cindy wins is $\frac{10}{32} = \frac{5}{16}$. The required sum is 21.

9. **803** The first few terms of the sequence are $a_1 = 1,\ a_2 = \frac{2}{3},\ a_3 = \frac{1}{2},\ a_4 = \frac{2}{5},\ a_5 = \frac{1}{3}\ ,\ a_6 = \frac{2}{7},$ $a_7 = \frac{1}{4},\ a_8 = \frac{2}{9}, \ldots$. Notice the even terms each have 2 in the numerator and $n+1$ in the denominator. This fact can be proven by mathematical induction.
Thus, $a_{800} = \frac{2}{801}$ and the required sum is 803.

10. **264** The planar cross-section required is the square base of an equilateral pyramid. The volume of this pyramid is $\frac{1}{2}$ the volume of the regular octahedron. So, $\frac{1}{3}Bh = 1010$, where B is the area of the pyramid's base and h is the height of the pyramid. If $x = AN = AD$, then $AO = \frac{x}{\sqrt{2}}$. By the Pythagorean Theorem in ΔAOD, $h^2 + \left(\frac{x}{\sqrt{2}}\right)^2 = x^2 \rightarrow$ $h^2 = \frac{x^2}{2} \rightarrow h = \frac{x}{\sqrt{2}}$. Since $B = x^2$, $\frac{1}{3}(x^2)\left(\frac{x}{\sqrt{2}}\right) = 1010 \rightarrow$ $x^3 = 3030\sqrt{2} \rightarrow x = 16.242 \ldots$. Thus $x^2 \approx 264$.



Team Problem Solving - NMT 2020 **Solutions**

1. **2** Since all primes are odd except for the number 2, the sum of an even number of primes will be even , unless one of them is the number 2.

2. **216** Let s be an edge of the cube. The surface area of the cube is $6s^2$ and the cube of the area is $(6s^2)^3 = 216s^6$. The volume of the cube is s^3 and the square of the volume is $(s^3)^2 = s^6$. Thus, $n = 216$.

3. **25** Start with the top row. Since 17 and 31 are prime factors, $17 \cdot 31 = 527$ is a factor. The other factor must be less than 10 and needs to result in a units digit of 6. The only possible factor is 8, and $8 \cdot 527 = 4216$. The missing digit in the top row is 1. Since 11 and 19 are prime factors of the middle number, $11 \cdot 19 = 209$ is also a factor. The remaining factor must result in a units digit of 4. Thus, the units digit of the remaining factor is 6. Try using the numbers 16, 26, 36,... until the product results in a 4-digit number ending in 24. The only factor that works is 36. Since $36 \cdot 209 = 7524$, the missing digits are 7 and 5. In the bottom row, 11, 43, and $11 \cdot 43 = 473$ are factors of the number. The product with the remaining factor must result in a units digit of 8, so it must have a units digit of 6. Try using the same numbers as in the middle row. Since $473 \cdot 16 = 7568$, the missing digits are again 7 and 5. Thus, $1 + 7 + 5 + 7 + 5 = 25$.

4	2	1	6
7	5	2	4
7	5	6	8

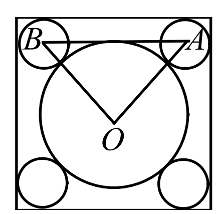
4. **49** Let x be the integer. Then, $x = \frac{343}{\sqrt{x}} \rightarrow x\sqrt{x} = 343 \rightarrow x^{\frac{3}{2}} = 343 \rightarrow x = 343^{\frac{2}{3}} = \left(\sqrt[3]{343}\right)^2 = 49$.

5. **23** Multiply the three given equations to get $(nm)(mt)(nt) = (48)(54)(72) = (6)(8)(6)(9)(8)(9)$. Thus, $nmt = (6)(8)(9)$. Since $nm = 48$, $t = 9$. Since $mt = 54$, $n = 8$. It follows that $m = 6$, so $n + m + t = 8 + 6 + 9 = 23$.

6. **96** Since $\measuredangle BAP \cong \measuredangle PAC \cong \measuredangle CAD$, the measures of the angles are each $30°$. In a $30°$-$60°$-$90°$ triangle, the ratios of the corresponding sides are $1 : \sqrt{3} : 2$. The sides of ΔDAC have lengths $CD = 6$, $AC = 12$, and $AD = 6\sqrt{3}$. The sides of ΔABP have lengths $AB = 6$, $BP = 2\sqrt{3}$ and $AP = 4\sqrt{3} = PC$. The perimeter of ΔPAC is $12 + 4\sqrt{3} + 4\sqrt{3} = 12 + 8\sqrt{3}$. Thus, $12(8) = 96$.

7. **247** In order to minimize the difference, the two integers must be as close as possible. If the leftmost digit of the minuend is 5, the leftmost digit of the subtrahend is 4. Then, minimize the minuend to get 50123, and maximize the subtrahend to get 49876. The difference is $50123 - 49876 = 247$.

8. **52** Connect the center of the big circle to the centers of two consecutive small circles. Since $OA = OB = 4$, $AB = 4\sqrt{2}$ and the length of a side of the square is $2 + 4\sqrt{2}$. The area of the square is $\left(2 + 4\sqrt{2}\right)^2 = 4 + 16\sqrt{2} + 32 = 36 + 16\sqrt{2}$. Thus, $a = 36$, $b = 16$, and $a + b = 52$.



9. **47** Square both sides of the given equation and simplify: $\left(a + \frac{1}{a}\right)^2 = 9 \rightarrow a^2 + 2 + \frac{1}{a^2} = 9 \rightarrow a^2 + \frac{1}{a^2} = 7$. Square again: $\left(a^2 + \frac{1}{a^2}\right)^2 = a^4 + 2 + \frac{1}{a^4} = 49 \rightarrow a^4 + \frac{1}{a^4} = 47$.

10. **7** Rewrite the fraction in expanded form and multiply by $\frac{10,000}{10,000}$:
$\frac{(d.5)^4}{(0.d5)^2} = \frac{(d.5)(d.5)(d.5)(d.5)}{(0.d5)(0.d5)} = \frac{(d5)(d5)(d5)(d5)}{(d5)(d5)} = (d5)^2 = 5625.$ Then, $d5 = 75 \rightarrow d = 7.$

11. **36** Let x be the distance between the parallel lines. The area of parallelogram $AECF$ can be calculated in two ways: $(AE)(BC)$ or $(EC)(x)$. Thus, $(45)(120) = 150x \rightarrow x = \frac{(45)(120)}{150} = 36.$

12. **30** If $12,\ b,\ c$ form an arithmetic sequence with a common difference, d, then $b = 12 + d$ and $c = 12 + 2d$. So, in terms of d, the geometric sequence is $12,\ 15 + d,\ 30 + 2d$. By the definition of a geometric sequence, $\frac{15+d}{12} = \frac{30+2d}{15+d} \rightarrow \frac{15+d}{12} = 2 \rightarrow 15 + d = 24 \rightarrow d = 9.$ Thus, $c = 12 + 2(9) = 30.$ [Note: $\frac{30+2d}{15+d} = 2,$ provided $d \neq -15.$]

13. **108** The three possible means and medians are 37, 56, and a number between 37 and 56. The sum of the five numbers equals 5 times the mean of the numbers. If the mean and median are 37, then $x + 22 + 37 + 56 + 89 = 5(37) \rightarrow x + 204 = 185 \rightarrow x = -19.$ If the mean and median are 56, then $22 + 37 + 56 + x + 89 = 5(56) \rightarrow x + 204 = 280 \rightarrow x = 76.$ If the mean and median are between 37 and 56, then $22 + 37 + x + 56 + 89 = 5x \rightarrow 4x = 204 \rightarrow x = 51.$ Thus, the required sum is $-19 + 76 + 51 = 108.$

14. **25** The average weight of all of the dogs and cats is $\frac{(5)(28)+(3)(20)}{5+3} = \frac{140+60}{8} = 25.$

15. **17** Let $\frac{x}{2} = 2w \rightarrow x = 4w.$ So, $f(2w) = (4w)^2 - (4w) + 12 = 14 \rightarrow 16w^2 - 4w - 2 = 0 \rightarrow$ $8w^2 - 2w - 1 = 0 \rightarrow (4w + 1)(2w - 1) = 0 \rightarrow w = -\frac{1}{4}$ or $w = \frac{1}{2}$. Thus, $k = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$ and $68k = 17.$

16. **32** The given equation can be written as $\tan 5x = \sqrt{3} \rightarrow 5x = \frac{\pi}{3} + \pi k$, where k is an integer. So, $x = \frac{\pi}{15} + \frac{\pi k}{5}$ which, on the interval $[0, 2\pi]$ yields the ten roots:
$\left\{\frac{\pi}{15}, \frac{4\pi}{15}, \frac{7\pi}{15}, \frac{10\pi}{15}, \frac{13\pi}{15}, \frac{16\pi}{15}, \frac{19\pi}{15}, \frac{22\pi}{15}, \frac{25\pi}{15}, \frac{28\pi}{15}\right\}$. The sum of these roots is $\frac{145\pi}{15} = \frac{29\pi}{3}$.
Thus, $29 + 3 = 32.$

17. **91** Use the theorem that states that tangent segments to a circle from an external point are congruent. Therefore, $AD = AF = 63.$ If $x = FE = EC,$ then $BE = 63 - x.$ Use the Pythagorean Theorem in ΔABE to solve for x: $(63 - x)^2 + 84^2 = (63 + x)^2 \rightarrow$ $63^2 - 126x + x^2 + 84^2 = 63^2 + 126x + x^2 \rightarrow 252x = 7056 \rightarrow x = 28.$ Thus, $AE = 63 + 28 = 91.$

18. **6** The units digits for each of the numbers, 2, 3, and 7, when raised to a power create patterns. The units digit pattern for powers of 2 is 2-4-8-6. The units digit pattern for powers of 3 is 3-9-7-1. The units digit pattern for powers of 7 is 7-9-3-1. Since each pattern repeats after 4 digits and since 2020 is divisible by 4, the units digit of 3^{2020} is 1. The units digit of 2^{2019} is 8, and the units digit of 7^{2021} is 7. The product of these three numbers is (8)(1)(7)=56 which has a units digit of 6.

19. **1** Apply the base-change rule to each of the fractions in the given expression:

$$\frac{1}{\log_2 10!}+\frac{1}{\log_3 10!}+\frac{1}{\log_4 10!}+\cdots+\frac{1}{\log_{10} 10!}=\frac{\log 2}{\log 10!}+\frac{\log 3}{\log 10!}+\frac{\log 4}{\log 10!}+\cdots+\frac{\log 10}{\log 10!}.$$

Now apply the rule $\log a+\log b=\log ab$:

$$\frac{\log 2}{\log 10!}+\frac{\log 3}{\log 10!}+\frac{\log 4}{\log 10!}+\cdots+\frac{\log 10}{\log 10!}=\frac{\log(2\cdot 3\cdot 4\cdot \ldots \cdot 10)}{\log 10!}=\frac{\log 10!}{\log 10!}=1.$$

20. **182** The number of ways of choosing 4 of the 16 points in the grid is ${}_{16}C_4=\frac{16!}{12!4!}=1820$. Since there are 10 sets of points that lie on a line (4 rows, 4 columns, and 2 diagonals), the probability of choosing one of them is $\frac{10}{1820}=\frac{1}{182}$. The reciprocal is 182.

Tie Breakers - NMT 2020 **Solutions**

1.

2.

3.