Grade Level 9 - NMT 2019 **Solutions**

1. **12** If $A = \pi r^2 = 36\pi$, then $r^2 = 36 \longrightarrow r = 6$, and $d = 12$.

2. **19** The slope of the line is $\frac{-10-(-2)}{79-31} = -\frac{1}{6}$ and an equation of the line is $y + 2 = -\frac{1}{6}(x - 31)$. To find the x-intercept, let $y = 0$. So, $2 = -\frac{1}{6}(x - 31) \longrightarrow 12 = -x + 31 \longrightarrow x = 19$.

3. **35** By simplifying the radicals, $\sqrt{72} + 3\sqrt{162} = 6\sqrt{2} + 3(9\sqrt{2}) = 33\sqrt{2}$. Thus, $33 + 2 = 35$.

4. **0** If $abcd = 0$ and $abde = 1$, then $c = 0$ and $0(a + b + c + d + e) = 0$.

5. **96** The least sum of all 10 test scores is $10 \cdot 80 = 800$. The sum of the first 7 test scores is $7 \cdot 72 = 504$. So, the least sum of the next 3 test scores is $800 - 504 = 296$. Thus, 2 of these scores must each be 100 and the minimum score possible is 96.

6. **4** The decimal representation of $\frac{5}{13}$ is $0.\overline{384615}$ which has a repetend of length 6. Since $\frac{2019}{6} =$ 336 r 3, the 2019th digit after the decimal point is the same as the 3rd digit, i.e. 4.

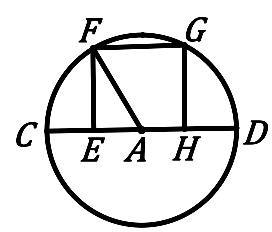
7. **101** Let x and $x + 2$ represent the lengths of the other leg and hypotenuse, respectively. By the Pythagorean Theorem, $(x + 2)^2 = 20^2 + x^2 \longrightarrow x^2 + 4x + 4 = 400 + x^2 \longrightarrow 4x = 396 \longrightarrow x = 99$. Thus, $x + 2 = 101$.

8. **16** The lines intersect, in pairs, at the points $(0, 5)$, $(4, 1)$ and $(-4, 1)$ forming a triangle whose base has length 8 and whose height has length 4. Thus, $A = \frac{1}{2}bh = \frac{1}{2}(8)(4) = 16$.

9. **5** If $f(x) = x^2 + 5x - 36$, then $f(2a - 1) = (2a - 1)^2 + 5(2a - 1) - 36 = 0 \longrightarrow 4a^2 - 4a + 1 + 10a - 5 - 36 = 0 \longrightarrow 4a^2 + 6a - 40 = 0 \longrightarrow 2a^2 + 3a - 20 = 0 \longrightarrow (2a - 5)(a + 4) = 0 \longrightarrow a = \frac{5}{2}$ (positive value only). Thus, $2a = 5$.

10. **23** Together, the two trains travel 60 miles each hour, or 1 mile each minute. Therefore, they will be 23 miles apart 23 minutes before they pass each other.

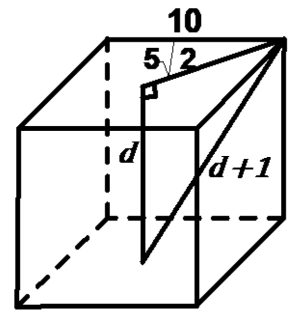
11. **20** If $CD = 10$, then radius $AF = 5$. If $EH = EF = 2x$, then $EA = x$. By the Pythagorean Theorem, $x^2 + (2x)^2 = 5^2 \longrightarrow 5x^2 = 25 \longrightarrow x = \sqrt{5}$ and $2x = 2\sqrt{5}$. Thus, the area of the square is $(2\sqrt{5})^2 = 20$.



12. **119** The LCM of 8, 6, 4, and 2 is 24. The number we seek is 1 less than a multiple of 24 (23, 47, 71, 95, 119,…) and 2 less than a multiple of 11 (31, 42, 53, 64, 75, 86, 97, 108, 119,…). Thus, 119 satisfies both conditions.

13. **5** It is given that $\frac{x+y}{2} = z - 3$, $\frac{x+z}{2} = y - 6$, and $\frac{y+z}{2} = 3 - x$. Multiplying by 2 and rearranging terms yields $x + y - 2z = -6$, $x - 2y + z = -12$, and $2x + y + z = 6$. Add the equations to get $4x = -12 \longrightarrow x = -3$. By substitution in the first and third equations above, $y - 2z = -3$ and $y + z = 12$. Subtract these equations to get $3z = 15 \longrightarrow z = 5$.

14. **721** Select 6 slips of paper. This can be done in ${}_{20}C_6$ ways. There is only one way to arrange each of the selections in ascending order. Since there are ${}_{20}P_6$ arrangements, the required probability is $\frac{{}_{20}C_6}{{}_{20}P_6} = \frac{1}{6!} = \frac{1}{720}$. Thus, $1 + 720 = 721$. Alternatively, for every ${}_6P_6 = 6!$ arrangements of the papers selected, there is only 1 arrangement that is in the proper order. Thus, the probability is $\frac{1}{6!} = \frac{1}{720}$. So, $1 + 720 = 721$.

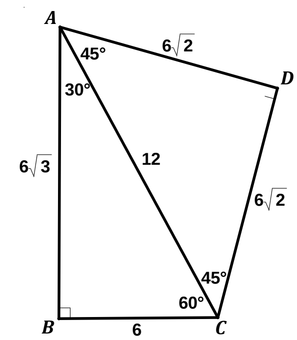


15. **245** Let d and $d + 1$ be the depth of the pond and the length of the reed, respectively. The distance from the center of the pond's surface to a corner is $\frac{1}{2}(10\sqrt{2}) = 5\sqrt{2}$. By the Pythagorean Theorem, $d^2 + (5\sqrt{2})^2 = (d + 1)^2 \longrightarrow d^2 + 50 = d^2 + 2d + 1 \longrightarrow 2d = 49 \longrightarrow d = 24.5$. Thus, $10d = 245$.

Grade Level 10 - NMT 2019 **Solutions**

1. **5** The slope of $\overline{AC}$ is $-\frac{2}{3}$. Therefore, the slope of the altitude to $\overline{AC}$ is $\frac{3}{2}$. The required sum is 5.

2. **13** In order to compute the value of $f(11)$, it is useful to compute the value of x. From $2x-1=11$, $x=6$. Therefore, $f(11)=6^2-4\cdot 6+1=13$.



3. **57** Using the given side lengths, we see that ΔADC is a 45-45-90 triangle and ΔABC is a 30-60-90 triangle. So, $[ABCD]$, the area of quadrilateral $ABCD$ is $[ABC]+[ADC]=\frac{1}{2}(6)(6\sqrt{3})+\frac{1}{2}(6\sqrt{2})^2=18\sqrt{3}+36$. Thus, $18+2+36=57$.

4. **272** Using the fact that the sum of the roots is $-a$, $2a+99=-a\rightarrow 3a=-99\rightarrow a=-33$. So, the roots are $-33+49=16$ and $-33+50=17$. Since b is the product of the roots, $b=16\cdot 17=272$.

5. **6** In a regular polygon, an interior angle and an exterior angle are supplementary. So, if the sum of the measures of two angles is $180°$ and their difference is $60°$, then the two angles measure $120°$ and $60°$. The greater value of n, the number of sides of the polygon, is found when the exterior angle's measure is $60°$. Thus, $n=\frac{360}{60}=6$.

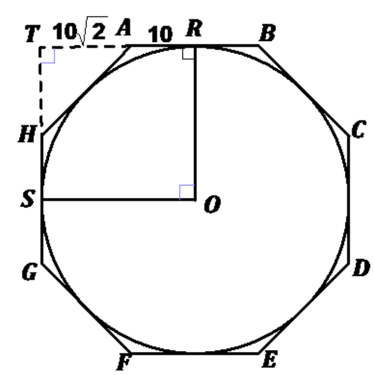
6. **56** If the point (a,b) is the center of dilation by a factor of 2, the given point, $(5,2)$, is the midpoint of the segment joining the center of dilation and the image $(2,-3)$. So, $\frac{a+2}{2}=5\longrightarrow a=8$, and $\frac{b-3}{2}=2\longrightarrow b=7$. Thus, $ab=56$.

7. **49** Since all spheres are similar, the ratio of the lengths of their radii is the square root of the ratio of their areas. So, $\frac{a}{b}=\sqrt{\frac{539}{44}}=\sqrt{\frac{49}{4}}=\frac{7}{2}$. Thus, $7^2=49$.

8. **55** Average rate of speed may be expressed as total distance traveled divided by total time traveled. Let the total distance Claire runs be d. Therefore, her average speed may be expressed as $\frac{d}{\frac{d/4}{12}+\frac{3d/4}{6}}=\frac{48}{7}$. The required sum is 55.

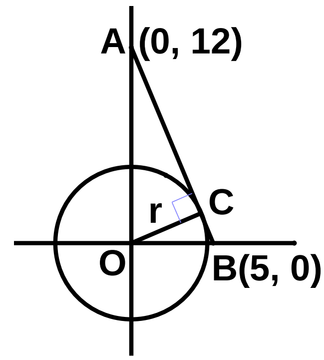
9. **15** In a circle, the length of an arc, s, is equal to θr, where θ is the measure of the arc's central angle in radians, and r is the length of the radius. So, $s=1.5r$. Also, $2r+s=35\longrightarrow 2r+1.5r=35\longrightarrow 3.5r=35\longrightarrow r=10$. Thus, $s=(1.5)(10)=15$.

10. **22** In octagon $ABCDEFGH$, extend $\overline{BA}$ through point A, and extend $\overline{GH}$ through point H to meet at point T. Exterior angles of the polygon, THA and TAH, each measure $45°$, so ΔHAT is an isosceles right triangle where $HA=20$, and $HT=TA=10\sqrt{2}$. Since $\overline{OR}$ bisects $\overline{AB}$, $AR=10$, and since quadrilateral $TSOR$ is a square, $TR=OS=10+10\sqrt{2}$. Thus, $10+10+2=22$.



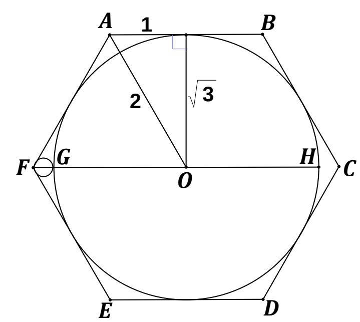
11. **40** The given line intersects the axes at the points $(0, -b)$ and $\left(\frac{b}{3}, 0\right)$. So, the area of the triangle, $1200 = \frac{1}{2}(b)\left(\frac{b}{3}\right) \longrightarrow 1200 = \frac{b^2}{6} \longrightarrow b^2 = 7200 \longrightarrow b = 60\sqrt{2}$. Thus, the x-intercept is $\frac{60\sqrt{2}}{3} = 20\sqrt{2}$, and $(20)(2) = 40$.

12. **73** The x- and y-intercepts of the given line and the origin determine right ΔAOB with vertices as shown in the diagram. Radius $\overline{OC} \perp \overline{AB}$, so $\overline{OC}$ is an altitude of ΔAOB. By the Pythagorean Theorem, the length of the hypotenuse of ΔAOB is 13. So, the area of $\Delta AOB = \frac{1}{2}(5)(12) = \frac{1}{2}(13)(r) \longrightarrow r = \frac{60}{13}$ and $60 + 13 = 73$.



13. **8** Since the question asks for a term not containing a z, we need expand only $(x+y)^2(2x+3y) = (x^2+2xy+y^2)(2x+3y) =$
$2x^3 + 3x^2y + 4x^2y + 6xy^2 + 2xy^2 + 3y^3 = 2x^3 + 7x^2y + 8xy^2 + 3y^3$.
The required coefficient is 8.

14. **14** Since all circles are similar, the ratio of the lengths of the radii equals the ratio of the lengths of any two corresponding parts of the circles. Draw $\overline{FH}$ from a vertex of the hexagon to a point on the larger circle passing through point G, the circles' point of tangency, and the larger circle's center. If we consider the dilation with center F that takes point H to point G, that dilation transforms the larger circle to the smaller circle. So, the scale factor of the dilation, $\frac{FH}{FG}$, equals the ratio of the lengths of the radii. If we let a side of the hexagon have length 2, $AO = FO = 2$ and $GO = HO = \sqrt{3}$. So, $FH = FO + HO = 2 + \sqrt{3}$, and $FG = FO - GO = 2 - \sqrt{3}$. Thus, the ratio of the radii is $\frac{2+\sqrt{3}}{2-\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{7+4\sqrt{3}}{4-3} = 7 + 4\sqrt{3}$. Thus, $7 + 4 + 3 = 14$.



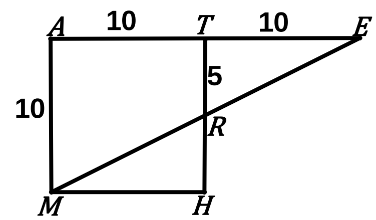
15. **11** Recognize that $br + b + r = br + b + r + 1 - 1 = (b+1)(r+1) - 1$. In order for $br + b + r$ to be divisible by 3, it is necessary that $(b+1)(r+1) - 1 \equiv 1 (\text{mod } 3)$. This can occur in two ways: Each of $b+1$ and $r+1$ must be congruent to 1 (mod3) or to 2(mod 3). If each of $b+1$ and $r+1$ is congruent to 1 (mod3), then each of b and r is congruent to 0(mod3). Thus there are four ordered pairs of (b, r) that satisfy the condition of the problem: $(3,3), (3,6), (6,3)$, and $(6,6)$. Similarly, if each of $b+1$ and $r+1$ is congruent to 2 (mod3), then each of b and r is congruent to 1(mod3). This yields an additional four ordered pairs of (b, r) that satisfy the condition of the problem: $(1,1), (1,4), (4,1)$, and $(4,4)$. The required probability is $\frac{8}{36} = \frac{2}{9}$. The required sum is 11.

Grade Level 11 - NMT 2019 **Solutions**

1. **41** The given expression $\left(\frac{(-3)^3}{2^6}\right)^{-2/3} = \left(\frac{-27}{64}\right)^{-2/3} = \left(\frac{64}{-27}\right)^{2/3} = \left(\sqrt[3]{\frac{64}{-27}}\right)^2 = \left(\frac{4}{-3}\right)^2 = \frac{16}{9}$.
Thus, $2p + q = 2 \cdot 16 + 9 = 41$.

2. **45** $f(2) = 25 \rightarrow g(25) = 5$ and $g^{-1}(3) = 9$. The required product is $5 \cdot 9 = 45$.

3. **2** Since the sum of the measures of the angles of a triangle is 180°, $6x = 180 \longrightarrow x = 30$. So, $(\sin 30°)^2 + (\sin 60°)^2 + (\sin 90°)^3 = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + 1^3 = \frac{1}{4} + \frac{3}{4} + 1 = 2$.



4. **75** Since $\overline{RT} \parallel \overline{MA}$ and $\overline{RT}$ contains the midpoint of $\overline{AE}$, $RT = 5$. The required area is the sum of the areas of a 10 by 5 rectangle and a triangle whose area is half the area of the rectangle. Thus, $50 + 25 = 75$. Alternatively, since $MATR$ is a trapezoid, its area is $\frac{1}{2}(10)(5+10) = 75$.

5. **5** According to the conditions in the problem, $\frac{2^{k+2} - 2^k}{k+2-k} = 48 \rightarrow 2^k(2^2 - 1) = 96 \rightarrow 2^k = 32 \rightarrow k = 5$.

6. **2** The key to solving this problem is to realize that the line of centers of the two circles is also the perpendicular bisector of $\overline{PQ}$. This can be proven by drawing two radii in each circle to the points P and Q and using double triangle congruence. The centers of the circles are $(-5, 4)$ and $(5, -1)$. The slope of the line of centers is $-\frac{1}{2}$. Thus the slope of $\overline{PQ}$ is 2, the opposite reciprocal of $-\frac{1}{2}$.

7. **60** Working alone, Sam can mow the lawn in x minutes and Todd can mow the lawn in $x - 20$ minutes. Working together, Sam's portion of the job is $\frac{24}{x}$, and Todd's portion is $\frac{24}{x-20}$. So, $\frac{24}{x} + \frac{24}{x-20} = 1 \rightarrow 24(x - 20 + x) = x(x - 20) \rightarrow x^2 - 68x + 480 = 0 \rightarrow (x - 60)(x - 8) = 0 \rightarrow x = 60$ (only).

8. **520** An equation of the type $a^c = 1$ is solved when $a = 1$ and c is any number, or when $c = 0$ and $a \neq 0$, or when $a = -1$ and c is even. Therefore, $x^2 - 11x + 29 = 1 \rightarrow x^2 - 11x + 28 = 0 \rightarrow (x-7)(x-4) = 0 \rightarrow x = 4$ or 7. Also, $x^2 - 9x + 18 = 0 \rightarrow (x-6)(x-3) = 0 \rightarrow x = 3$ or 6. Also, $x^2 - 11x + 29 = -1 \rightarrow x^2 - 11x + 30 = 0 \rightarrow (x-6)(x-5) = 0 \rightarrow x = 6$ or 5. So, the solution set is $\{3,4,5,6,7\}$. The product of these roots is 2520 and the required remainder upon division by 1000 is 520.

9. **11** The length of the main diagonal of the prism is calculated using the Pythagorean Theorem in three dimensions: $d^2 = \left(5\sqrt{2}\right)^2 + 12^2 + 13^2 = 363$. The main diagonal of the inscribed prism has the same length as the main diagonal of the inscribed cube. Therefore, if the length of a side of the cube is x, then $3x^2 = 363 \rightarrow x^2 = 121 \rightarrow x = 11$.

10. **48** If $N = p^a q^b r^c$, where p, q, and r are primes and $a, b,$ and c are non-negative integers, then the number of factors of N is $(a+1)(b+1)(c+1)$. The only positive integer products equal to 10 are 10 by 1 or 5 by 2. Either $a+1=10$ and $b+1=1$ or $a+1=5$ and $b+1=2$. In the first case, a = 9 and b = 0 yielding $2^{10} = 512$. But, we can do better with the second case. There, $a = 4$ and $b = 1$ yielding $2^4 \cdot 3^1 = 48$.

11. **36** The given equation is equivalent to $(1 + 2x + 4x^2 + 8x^3 + \cdots) = \frac{25}{3}$. The left side of the equation is the sum of an infinite geometric series whose first term is 1 and whose ratio is $2x$. So, the equation can be re-written as $\frac{1}{1-2x} = \frac{25}{3} \to x = \frac{11}{25}$. The required sum is $11 + 25 = 36$.

12. **9** An equation of the form: $a + \frac{1}{a} = \frac{10}{3} = 3 + \frac{1}{3}$ has the solution set $\left\{3, \frac{1}{3}\right\}$. We recognize $\log_3 x$ and $\log_x 3$ as reciprocals and apply the idea of the first sentence. So, $\log_3 x = 3$ or $\log_3 x = \frac{1}{3} \to x = 27$ or $x = \sqrt[3]{3}$. The sum of these roots is $27 + \sqrt[3]{3}$ and the required quotient is $\frac{27}{3} = 9$.

13. **42** We may re-write $g(x) = 5f\left(-\frac{1}{3}(x - 30)\right) + 1$ to see that the transformations that move $f(x) \to g(x)$ are a reflection over the y–axis, followed by a horizontal scaling of 3, followed by a translation 30 to the right, followed by a vertical scaling of 5, followed by a vertical translation of 1. Therefore, the point whose coordinates are $(-4,-1) \to (4,-1) \to (12,-1) \to (42,-1) \to (42,-5) \to (42,-4)$. So, the required x-coordinate is 42.

14. **225** The Rational Roots Theorem tells us that the possible rational roots of $f(x)$ are $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}$. Descartes Rule of Signs tells us that there are no positive rational roots. The product of the roots is $\frac{8}{8} = 1$ and since the roots are distinct, -1 cannot be a root and the roots must occur in reciprocal pairs. One of those pairs cannot be -8 and $-\frac{1}{8}$ because if -8 were a root, then -1 would also have to be a root. Therefore, the roots are $-4, -\frac{1}{4}, -2,$ and $-\frac{1}{2}$. Thus, we can rewrite $f(x) = (x+2)(x+4)(2x+1)(4x+1) \to f(1) = 225$.

15. **44** For the purposes of this solution, we will consider students A, B, C, D, and E and name tags numbered 1, 2, 3, 4, and 5. It would be considered correct if student A had name tag 1 and student B had name tag 2, etc. There are four ways that name tag 1 can go to the wrong student. First, consider the case where 1 goes to B and 2 goes to A. Then, there are only two ways that the remaining tags can go to the incorrect students: 4 to C, 5 to D, and 3 to E, or 5 to C, 3 to D, and 4 to E. Now, consider the case where 1 goes to B and 2 goes to C: 3 to A, 5 to D, and 4 to E, or 4 to A, 5 to D, and 3 to E, or 5 to A, 3 to D, and 4 to E. Consider the case where 1 goes to B and 2 to D: 3 to A, 5 to C, and 4 to E, or 5 to A, 4 to C, and 3 to E, or 4 to A, 5 to C, and 3 to E. Lastly, consider that 1 goes to B and 2 goes to E: 3 to A, 5 to C, and 4 to D, or 4 to A, 5 to C, and 3 to D, or 5 to A, 4 to C, and 3 to D. Thus, there are a total of 11 cases where everyone is tagged incorrectly in the case of 1 goes to B. Similarly, there are 11 additional cases when 1 goes to each of C, D, and E. Thus, there are 11 times 4 or 44 ways to distribute the tags in the required manner.

Grade Level 12 - NMT 2019 **Solutions**

1. **16** Let $b = f(a)$. Then, $f(b) = b \longrightarrow 144 - 8b = b \longrightarrow b = 16$. So, $f(a) = 16 \longrightarrow 144 - 8a = 16 \longrightarrow a = 16$. [Note: If we started with the fact that $f(a) = a$, the condition $f\big(f(a)\big) = f(a)$ would automatically follow.]

2. **500** $f'(x) = 1000x^{999} + 999x^{998} + 998x^{997} + \cdots + 1$, so $f'(-1) = -1000 + 999 - 998 + \cdots + 1$. Grouping these numbers in pairs yields the value -1 five hundred times, so $f'(-1) = -500$ and the answer is 500. Alternatively, using the formula for the sum of a finite geometric series, we can rewrite $f(x) = \frac{x^{1001}-1}{x-1}$ and then use the quotient rule.

3. **11** Factoring by grouping, $3x^3 - x^2 - 33x + 11 = 0 \longrightarrow x^2(3x-1) - 11(3x-1) = 0 \longrightarrow (3x-1)(x^2-11) = 0$ giving roots of $x = \frac{1}{3}$ and $x = \pm\sqrt{11}$. Since $\sqrt{11} > \frac{1}{3}$, the least and greatest of these roots are $-\sqrt{11}$ and $\sqrt{11}$, respectively. Thus, $\big|\big(-\sqrt{11}\big)\big(\sqrt{11}\big)\big| = 11$.

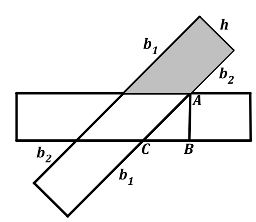
4. **0** Since $a_{2019} = a_1 r^{2018}$, then $34 = 17r^{2018} \longrightarrow 2 = r^{2018} \longrightarrow r = \pm\sqrt[2018]{2}$. So, $a_{1234} = 17r^{1233}$ which, since 1233 is odd, has two possible values which are negatives of each other. Therefore, the sum of all possible values of a_{1234} is 0.

5. **5** Since the curve is concave up, all tangent lines fall below the curve. Thus, the y-intercept of the tangent line must lie below the y-intercept of the parabola unless the tangent line is drawn <u>at</u> the y-intercept, (0, 5), of the parabola. Then, the tangent line and the curve have the same y-intercept. Therefore, the maximum value of the y-intercept is 5.

6. **1** Let $k = 6m + 3$ and $h = 8n + 4$, where m and n are integers. Then, $kh = (6m+3)(8n+4) = 48mn + 24m + 24n + 12 = 24(2mn + m + n) + 12$. This yields a remainder of 12 when divided by 24. So, there is only 1 possible remainder.

7. **2** Since x is approaching 1 from below, then $-1 < x < 1$ and the infinite geometric series given converges to $\frac{a}{1-r} = \frac{1}{1-x}$. So, $\lim\limits_{x \to 1^-} (1 - x^2)\left(\frac{1}{1-x}\right) = \lim\limits_{x \to 1^-} \frac{(1-x)(1+x)}{1-x} = 1 + 1 = 2$.

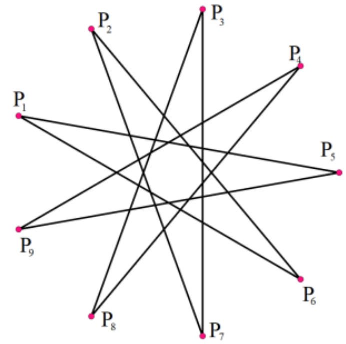
8. **982** The shaded quadrilateral is a trapezoid with height 2. Since ΔABC is an isosceles right triangle, $AC = 2\sqrt{2}$, and $b_1 + b_2 = 10 - 2\sqrt{2}$. So, the area of the trapezoid is $\frac{1}{2}(2)\big(10 - 2\sqrt{2}\big) = 10 - 2\sqrt{2}$.
Thus, $100(10) + 10(-2) + 2 = 982$.



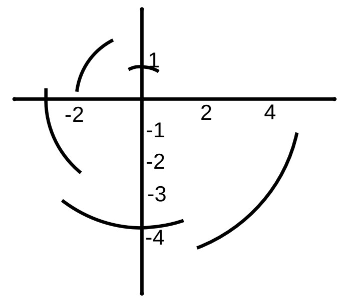
9. **525** Since $|3 - 4i| = 5$, multiplying a number by $3 - 4i$ will result in it being rotated by some (constant) angle about the origin and dilated by a factor of 5, centered at the origin. Since a rotation is a rigid motion, it preserves area. Dilating by a factor of 5, however, multiplies the area by 25. Since the area of the original triangle is $\frac{1}{2}(6)(7) = 21$, the area of the transformed triangle is $21 \cdot 25 = 525$.

10. **512** After the first transfer, the second container holds 1L of alcohol and 11L of water. Moving 2L back to the first container transfers $\frac{1}{12}(2) = \frac{1}{6}$L and $\frac{11}{12}(2) = \frac{11}{6}$L of water, giving the first container $4 + \frac{1}{6} = \frac{25}{6}$L of alcohol. This makes up $\frac{25/6}{10} = \frac{5}{12}$ of the solution.
Thus, $100(5) + 12 = 512$.

11. **27** Consider any one segment of the star, such as $\overline{P_1P_5}$. Each of P_2, P_3, and P_4 is the endpoint of two segments that intersect $\overline{P_1P_5}$, so each segment is crossed by other segments 6 times. Since there are 9 segments in the enneagram, this gives us 54 crossings. However, each point of intersection has been counted twice (once as the crosser and once as the cross-ee), so there are 27 points of intersection in total.

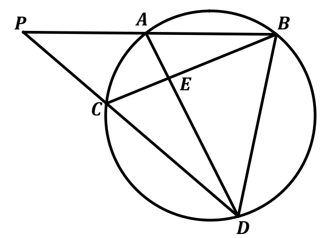


12. **512** Consider the 1st and 2nd numbers. They must be distinct and it is equally likely that either is larger, so there is a $\frac{1}{2}$ chance that the 1st number is larger than the 2nd number. The same thing, though, is true of the 3rd and the 4th, the 5th and the 6th, and so on. So, the probability that every pair of integers has the larger number on the left is $\left(\frac{1}{2}\right)^9 = \frac{1}{512}$. Thus, $p = 512$.

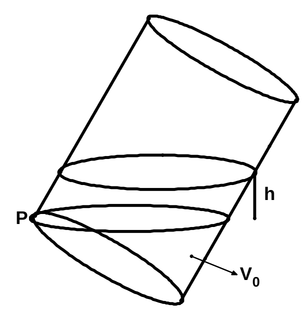


13. **15** Each arc shown in the graph has a measure of 1 radian and since, in radians, $s = r\theta$, each arc has a length equal to the length of its radius. Therefore, the total length of the curve is $1 + 2 + 3 + 4 + 5 = 15$.

14. **35** Since $\measuredangle BAD$ and $\measuredangle BCD$ are inscribed angles intercepting the same arc, they are congruent. Since $\measuredangle PAE$ and $\measuredangle PCE$ are supplementary to congruent angles, they are also congruent. But, since quadrilateral $PAEC$ is a cyclic quadrilateral, its opposite angles are supplementary, so that $\measuredangle PAE$ and $\measuredangle PCE$ are right angles, as are $\measuredangle BAD$ and $\measuredangle BCD$. So, $\overline{BD}$ is a diameter of the circle and arcBD is a semicircle. Thus, the circumference of the circle is $2(55) = 110$ and the diameter is $\frac{110}{\pi} \sim \frac{110}{\frac{22}{7}} = \frac{110 \cdot 7}{22} = 35$. [Note: Other approximations of π lead to the same answer].



15. **23** Let P be the highest point of the base of the glass. Let V_0 be the portion of the volume of water below point P (this volume is a constant once the water passes point P). The portion of the water between the level of point P and the surface is a non-right elliptical prism with a height of h. The elliptical surface has a minor axis of 6 and a major axis given by $\frac{6}{\cos 30°} = \frac{12}{\sqrt{3}}$. Since the area of an ellipse is given by πab, the area of the base of the prism is $\pi \cdot 3 \cdot \frac{6}{\sqrt{3}} = \frac{18\pi}{\sqrt{3}}$. Thus, the volume of water in the glass is given by $V = V_0 + \frac{18\pi}{\sqrt{3}} h$. Taking $\frac{d}{dt}$ of both sides yields $\frac{dV}{dt} = \frac{18\pi}{\sqrt{3}} \frac{dh}{dt} \longrightarrow 360 = \frac{18\pi}{\sqrt{3}} \frac{dh}{dt} \longrightarrow \frac{dh}{dt} = \frac{360\sqrt{3}}{18\pi} = \frac{20\sqrt{3}}{\pi}$. Thus, $20 + 3 = 23$



1. **5** Let G, B, M, and E represent the costs of the geography, biology, mathematics, and economics books, respectively. Then, $G + B = M,\ B = G + E,$ and $3E = 2M$. So, $E = \frac{2}{3}M$ and $B = G + \frac{2}{3}M \longrightarrow B = G + \frac{2}{3}(G + B) \longrightarrow B = 5G$. Thus, a biology book costs 5 times as much as a geography book.

2. **9** Since $\frac{673}{3333} = \frac{2019}{9999}$, the given fraction can be written as the repeating decimal $0.\overline{2019}$ which has a repetend of length 4. Thus, the 200th digit after the decimal point is the same as the 4th digit, i.e. 9.

3. **18** If x represents the number of diners, then $\frac{612}{x} = \frac{612}{x-1} + 2 \longrightarrow x^2 - x - 306 = 0 \longrightarrow (x - 18)(x + 17) = 0 \longrightarrow x = 18,$ only.

4. **55** If $\frac{7x-4y}{2x+y} = 7$, then $7x = -11y \longrightarrow y = -\frac{7}{11}x$. So, $\frac{13x-11y}{x+y} = \frac{13x+7x}{x-\frac{7}{11}x} = \frac{20x}{\frac{4}{11}x} = 55.$

5. **104** The given lines intersect at $(-10, 2)$. So $(-10, 2)$ is the center of the circle and $(x + 10)^2 + (y - 2)^2 = 16$ is an equation of the circle. Expanding, we get $x^2 + y^2 + 20x - 4y + 88 = 0$. Thus, $a + b + c = 20 - 4 + 88 = 104$.

6. **825** R is an infinite geometric series whose sum is $\frac{a}{1-r} = \frac{3}{1-\frac{1}{3}} = \frac{9}{2}$. The number of terms of arithmetic series T is found by $a_n = a_1 + d(n - 1) \longrightarrow 100 = -50 + 3(n - 1) \longrightarrow n = 51$. So, the sum of series T is $\frac{n}{2}(a_1 + a_n) = \frac{51}{2}(-50 + 100) = 1275$. Thus, $T - 100R = 1275 - 100\left(\frac{9}{2}\right) = 1275 - 450 = 825$.

7. **200** Emma and Ray pass each other for the second time on their return run toward their original starting points. On this return run, assume that they meet x feet from Ray's original location. Then, $\frac{1800+1800-x}{15} = \frac{1800+x}{12} \longrightarrow x = 600$. So, the time it takes to meet 600 feet from Ray's original starting point is $\frac{1800+1800-600}{15} = 200$ seconds.

8. **20** The prime factorization of $20!$ is $20! = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. Therefore, $20!$ has $19 \cdot 9 \cdot 5 \cdot 3 \cdot 2^4$ factors. [Note that if $N = p^r \cdot q^s$ where p and q are prime, N has $(r + 1)(s + 1)$ factors.] An odd factor of $20!$ has no positive integral powers of 2 as a factor. So, the number of odd factors of $20!$ is $1 \cdot 9 \cdot 5 \cdot 3 \cdot 2^4$. Thus, the probability of choosing an odd factor is $\frac{1\cdot 9\cdot 5\cdot 3\cdot 2^4}{19\cdot 9\cdot 5\cdot 3\cdot 2^4} = \frac{1}{19}$ and $p + q = 20$.

9. **239** Since the plane cuts the solid so that the smaller pyramid and the given pyramid are similar solids, and since $\frac{\text{volume of smaller pyramid}}{\text{volume of given pyramid}} = \frac{1}{2}$, then, $\frac{\text{area of smaller pyramid's base}}{\text{area of given pyramid's base}} = \left(\sqrt[3]{\frac{1}{2}}\right)^2 = \frac{1}{\sqrt[3]{4}}$. If x is the area of the smaller pyramid's base, then $\frac{x}{19\cdot 20} = \frac{1}{\sqrt[3]{4}} \longrightarrow x \approx 239$.

10. **24** Let $x =$ the number of lawyers in the firm.

	# Lawyers working on larger box	# Lawyers working on smaller box
Day 1-first half	x	
Day 1-second half	x	
Day 2-first half	x	
Day 2-second half	$\frac{1}{2}x$	$\frac{1}{2}x$
Day 3-first half		8
Day 3-second half		8

The chart represents the number of lawyers working on each box during each half-day of the project. Since the larger box is 3 times the size of the smaller box, we have
$\frac{7}{2}x = 3\left(\frac{1}{2}x + 16\right) \longrightarrow x = 24.$

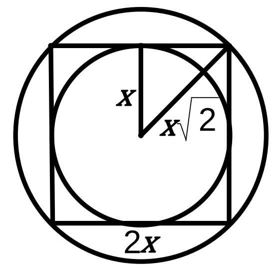
Team Problem Solving - NMT 2019 **Solutions**

1. **63** Since $f(a) = a^2 - 7a + a = a^2 - 6a = -9$, we know that $a^2 - 6a + 9 = 0$. Factor and solve to get $(a-3)^2 = 0$, so $a = 3$. It follows that $f(12) = 12^2 - 7(12) + 3 = 144 - 84 + 3 = 63$.

2. **169** The only integers that have exactly 3 divisors are squares of prime numbers. Since $13^2 = 169$ and since $17^2 > 200$, the answer is 169 and its divisors are 1, 13, and 169.

3. **6** Since the sum is 51, the middle number, $b = 17$. The primes less than 17 are 2, 3, 5, 7, 11, and 13. When a equals one of these, it is easy to find c. The only ones that result in c being prime are 3, 5, and 11. So, the ordered triples are the increasing sequences (3, 17, 31), (5, 17, 29), (11, 17, 23), and their corresponding decreasing sequences. .

4. **2** Notice that the diameter of the circumscribed circle is the diagonal of the square and that the length of the diameter of the inscribed circle is equal to the length of the side of the square. If we let the length of the side of the square be $2x$, the area of the inscribed circle is πx^2. The length of the diagonal of the square is $2x\sqrt{2}$, so the area of the circumscribed circle is $2\pi x^2$. Thus, the ratio we seek is $\frac{2\pi x^2}{\pi x^2} = 2$.



5. **5** Adding the two given equations yields $x(x+y) + y(x+y) = 25$. By GCF factoring, $(x+y)(x+y) = (x+y)^2 = 25$. Thus, the positive value of $x + y$ is 5.

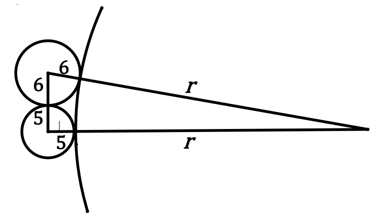
6. **0** Divide both sides of the given equation by 5 to get $5^x + 5^{-x} = 5^{-1} + 5^1$. Thus, by inspection, $x = 1$ or $x = -1$ and $-1 + 1 = 0$. Alternatively, let $u = 5^x \longrightarrow 5\left(u + \frac{1}{u}\right) = 26 \longrightarrow 5u^2 + 5 = 26u \longrightarrow 5u^2 - 26u + 5 = 0 \longrightarrow (5u-1)(5u-5) = 0 \longrightarrow u = \frac{1}{5}$ or $u = 5$. So, $5^x = \frac{1}{5}$ or $5^x = 5$ and $x = \pm 1$ which sum to 0.

7. **61** The y-intercept of the line defined by $3y + 8 = 5x$ is $-\frac{8}{3}$, so the line we seek is $y = -\frac{2}{5}x - \frac{8}{3}$. Multiply by 15 and arrange the terms to get $6x + 15y + 40 = 0$. Thus, $6 + 15 + 40 = 61$.

8. **85** The length of a diagonal of a rectangular prism can be found using the Pythagorean quadruple formula $a^2 + b^2 + c^2 = d^2$. Therefore, $d^2 = 5^2 + 12^2 + 84^2 = 13^2 + 84^2 = 85^2$. So, $d = 85$. [Note: Pythagorean triples 5-12-13 and 13-84-85 were used in this solution.]

9. **10** Square both sides of the equation $\sqrt{x + \sqrt{x+26}} + \sqrt{x - \sqrt{x+26}} = 6$ to get $x + \sqrt{x+26} + 2\sqrt{\left(x + \sqrt{x+26}\right)\left(x - \sqrt{x+26}\right)} + x - \sqrt{x+26} = 36 \longrightarrow 2x + 2\sqrt{x^2 - (x+26)} = 36 \longrightarrow \sqrt{x^2 - (x+26)} = 18 - x$. Squaring both sides again, $x^2 - (x+26) = 324 - 36x + x^2 \longrightarrow 35x = 350 \longrightarrow x = 10$.

10. **55** Let r be the length of the third radius. Applying the Pythagorean Theorem, $(r+6)^2 = (r+5)^2 + 11^2 \longrightarrow r^2 + 12r + 36 = r^2 + 10r + 25 + 121 \longrightarrow 2r = 110 \longrightarrow r = 55$. [Note: In the diagram, the (partially drawn) third circle must be the largest of the three circles because if the third circle is the smallest, then $(r+5)^2 + (r+6)^2 = 11^2$ yields a value of r that is irrational.]



11. **3** Find an approach to the solution by selecting a smaller number to examine: The divisors of 12 are 1, 2, 3, 4, 6, and 12. The sum of these divisors is 28. The sum of the reciprocals of the divisors is $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{12} = \frac{12+6+4+3+2+1}{12} = \frac{\text{Sum of divisors}}{\text{Original number}}$. So, for the given number 120, divide the sum of its divisors by the number to get $\frac{360}{120} = 3$.

12. **15** If $a_0 = 5$ and $a_1 = 7$, then $a_2 = \frac{3(7)+2(5)}{3} = \frac{31}{3}$ and $a_3 = \frac{3\left(\frac{31}{3}\right)+2(7)}{3} = \frac{45}{3} = 15$.

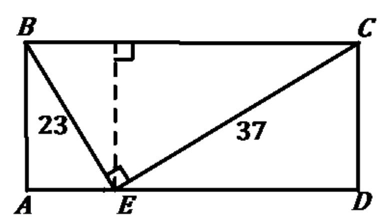
13. **39** Start with 1 and 2 down. The only perfect fourth powers that are 3-digit numbers are $4^4 = 256$ and $5^4 = 625$. Therefore, 1 across is either 2662 or 6226. Next consider 3 down. The Fibonacci number starting with 2 is 233 and starting with 6 is 610. Since the number must be even, 3 down is 610 and 1 across must be 2662. Since 2 across is a multiple of 401, with the first 3 digits 521, the units digit must be 3. The last box contains the number 1. The sum of these digits is 39.

1 **2**	2 **6**	3 **6**	4 **2**
5 **5**	**2**	**1**	**3**
6 **6**	**5**	**0**	**1**

14. **83** Let the original 2-digit number be n. Then, $10n + 7 = n + 754 \longrightarrow 9n = 747 \longrightarrow n = 83$.

15. **505** Since the sum is 726 and the mean of the numbers is 66, there are $\frac{726}{66} = 11$ numbers in the group. One of the numbers is 176, so the remaining 10 numbers have a sum of $726 - 176 = 550$. All the numbers are different and we want one number to be as great as possible. So, let the remaining 9 numbers be the least possible: $1+2+3+4+5+6+7+8+9 = 45$. Therefore, $550 - 45 = 505$, and this is the greatest possible number in the group.

16. **851** If BC and AB are, respectively, the lengths of the base and height of both the rectangle and triangle, then the area of ΔBEC is $\frac{1}{2}$ the area of rectangle $ABCD$. Since ΔBEC is a right triangle with legs 23 and 37, its area is $\left(\frac{1}{2}\right)(23)(37)$. Therefore, the area of rectangle $ABCD$ is $(23)(37) = 851$.



17. **28** Convert all of the logs to base-10 logs and apply the rules of logarithms: $\left(\frac{4\log 2}{\log 5}\right)x + \left(\frac{\log 7}{\log 3}\right)y = \frac{3\log 2}{2\log 5}$ and $\left(\frac{3\log 3}{\log 7}\right)x - \left(\frac{\log 5}{\log 2}\right)y = \frac{5\log 3}{2\log 7}$. Multiply the first equation by $\frac{\log 5}{\log 2}$ and the second equation by $\frac{\log 7}{\log 3}$. The products are $4x + \left(\frac{\log 7}{\log 3}\right)\left(\frac{\log 5}{\log 2}\right)y = \frac{3}{2}$ and $3x - \left(\frac{\log 7}{\log 3}\right)\left(\frac{\log 5}{\log 2}\right)y = \frac{5}{2}$. Add these two equations to get $7x = 4 \longrightarrow x = \frac{4}{7}$. This fraction is fully reduced, so we can multiply $4 \cdot 7 = 28$.

18. **345** The solution is based upon the theorem that the sum of the lengths of the two shortest sides of a triangle is greater than the length of the third side. When $x + 18$ is the longest side, $8x - 9 > x + 18 \longrightarrow x > 3$. When $3x$ is the longest side, $6x + 9 > 3x \longrightarrow x > -3$. When $5x - 9$ is the longest side, $4x + 18 > 5x - 9 \longrightarrow x < 27$. So, the integer solutions for x are contained in the intersection of the solution sets of the three inequalities, i.e. $4, 5, 6, \dots, 26$. Using the formula for the sum of an arithmetic series, $S_{23} = \frac{23}{2}(4 + 26) = 345$.

19. **52** Divide both sides of the given equation by $\cos x$ to get $\tan x = 2 \longrightarrow x = \tan^{-1} 2$. Using right triangle trig, $\sin(\tan^{-1} 2) = \frac{2}{\sqrt{5}}$ and $\cos(\tan^{-1} 2) = \frac{1}{\sqrt{5}}$. So, $\sin x \cos x = \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} = \frac{2}{5}$. Thus, $10(5) + 2 = 52$. [Note, if $x = \tan^{-1} 2 + k\pi$, where k is an odd integer, both $\sin x$ and $\cos x$ are negative and the final answer is unaffected].

20. **23** Since $f(x) = mx + b$, $f(f(x)) = m(mx + b) + b = m^2x + mb + b$. It follows that $f\left(f\left(f(x)\right)\right) = m(m^2x + mb + b) + b = m^3x + m^2b + mb + b = 64x + 399$. For these last two expressions to be equal for all x, $m^3 = 64$, so $m = 4$ and $4^2b + 4b + b = 399$. Thus, $21b = 399$ and $b = 19$. Consequently, $m + b = 4 + 19 = 23$.

Tie Breakers - NMT 2019 **Solutions**

1.

2.

3.