Grade Level 9 - NMT 2018 **Solutions**

1. **18** Represent the three given numbers as $3x, 4x$, and $5x$. Since the mean of these numbers is 24, $\frac{3x+4x+5x}{3} = 24 \longrightarrow 4x = 24 \longrightarrow x = 6$. Thus, the least of the three numbers is $3(6) = 18$.

2. **8** Substituting the given ordered pair into the equation yields $3(2) - b = -2 \longrightarrow b = 8$.

3. **512** Since the perimeter is 128, the base and height of the rectangle can be represented by x and $64 - x$. So, as a function of x, the area of the rectangle is $A(x) = x(64 - x) = -x^2 + 64x$. The maximum value of this quadratic function occurs at its turning point: $x = -\frac{b}{2a} = \frac{-64}{-2} = 32$. So, $A(32) = 1024$ and $\frac{1024}{2} = 512$.

4. **1** The sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is $-\frac{b}{a}$. So, for $9x^2 - 9x - 10 = 0$, $-\frac{b}{a} = \frac{9}{9} = 1$.
Alternatively, $9x^2 - 9x - 10 = 0 \longrightarrow (3x + 2)(3x - 5) = 0 \longrightarrow x = -\frac{2}{3}$ or $x = \frac{5}{3}$.
The sum of these roots is $-\frac{2}{3} + \frac{5}{3} = \frac{3}{3} = 1$.

5. **6** Let x represent the length of the second leg. So, $x^2 + \left(3\sqrt{6}\right)^2 = 9^2 \longrightarrow x^2 + 54 = 81 \longrightarrow x^2 = 27 \longrightarrow x = 3\sqrt{3}$. The required sum is 6.

6. **45** Let x and $60 - x$ represent the number of questions answered correctly and incorrectly, respectively. So, $x - 0.2(60 - x) = 42 \longrightarrow x - 12 + 0.2x = 42 \longrightarrow 1.2x = 54 \longrightarrow x = 45$.

7. **12** In an isosceles triangle, the altitude drawn to the base forms two congruent right triangles. The lengths of two pairs of congruent corresponding sides of these right triangles are 8 and 10. Using the Pythagorean triple 6-8-10, the altitude of the isosceles triangle has length 6, so the area of the isosceles triangle is $\frac{1}{2}(16)(6) = 48$. If x and $x - 10$ represent the lengths of the base and height of the rectangle, respectively, then $2x(x - 10) = 48 \longrightarrow x^2 - 10x - 24 = 0 \longrightarrow (x - 12)(x + 2) = 0 \longrightarrow x = 12$.

8. **500** If x represents the population at 12:01 AM on January 1, 2015, $x + 1200 - 0.11(x + 1200) = x - 32 \longrightarrow 0.89(x + 1200) = x - 32 \longrightarrow 0.89x + 1068 = x - 32 \longrightarrow 0.11x = 1100 \longrightarrow x = 10{,}000$. Thus, $\frac{10{,}000}{20} = 500$.

9. **20** There are $10!$ ways for 10 people to be arranged in a line. If the 2 adjacent boys count as one pair, they themselves can be arranged in $2!$ ways. So, together with the girls, the 1 pair of boys and the 8 girls can be arranged in $2! \cdot 9!$ ways. Thus, the required probability is $\frac{2! \cdot 9!}{10!} = \frac{2}{10} = 20\%$, and $p = 20$.

10. **909** The integer 10^{101} consists of 1 followed by 101 zeroes. If we subtract 1 from this integer, the result is an integer consisting of 101 9's. Thus, the sum of the digits is $9(101) = 909$.

11. **4** We solve the given inequalities by factoring: $x^2 - 2x - 8 \le 0 \longrightarrow (x + 2)(x - 4) \le 0 \longrightarrow -2 \le x \le 4$. The integers of this solution are: $\{-2, -1, 0, 1, 2, 3, 4\}$. Similarly, $x^2 - 2x - 3 \ge 0 \longrightarrow (x + 1)(x - 3) \ge 0 \longrightarrow x \le -1$ or $x \ge 3$. The integers of this solution that are common to the first integer set are: $\{-2, -1, 3, 4\}$. The sum of these integers is 4.

12. **25** Let $k = 100{,}000$. So $\frac{8^{10k}+4^{10k}}{8^{4k}+4^{11k}} = \frac{2^{30k}+2^{20k}}{2^{12k}+2^{22k}} = \frac{2^{20k}(2^{10k}+1)}{2^{12k}(1+2^{10k})} = 2^{8k} = 256^k = 256^{4000a}$.
So, $k = 4000a \longrightarrow 100{,}000 = 4000a \longrightarrow a = 25$.

13. **25** In $\frac{3mr-nt}{7mr-4nt}$, divide the numerator and denominator by nt: $\frac{\frac{3mr}{nt}-\frac{nt}{nt}}{\frac{7mr}{nt}-\frac{4nt}{nt}} = \frac{3\left(\frac{m}{n}\right)\left(\frac{r}{t}\right)-1}{7\left(\frac{m}{n}\right)\left(\frac{r}{t}\right)-4} =$
$\frac{3\left(\frac{4}{3}\right)\left(\frac{9}{14}\right)-1}{7\left(\frac{4}{3}\right)\left(\frac{9}{14}\right)-4} = \frac{\frac{18}{7}-1}{6-4} = \frac{\frac{11}{7}}{2} = \frac{11}{14}$. Thus, the required sum is 25.

14. **2** If the 5-digit number using the given digits starts with 1, there are 4! or 24 ways of arranging the other 4 digits. Similarly, there are 24 ways of forming 5-digit numbers that start with 2 or that start with 3. So far, we have counted 72 5-digit numbers. Within the next group of 24 5-digit numbers is the 86th number. Starting with 41… and starting with 42… there are 3! or 6 5-digit numbers. So now we have counted $72 + 12 = 84$ numbers. Starting with 431… there are 2 5-digit numbers: 43125 and 43152. Thus, the rightmost digit of the 86th number is 2.

15. **8** If we take away all of the outer cubes, we are left with a cube whose edge-length is $n-2$. So, there are $(n-2)^3$ unpainted cubes. Of the cubes that are removed, the corner cubes and those that border an edge of the original cube have more than one painted face. . So, the number of cubes with exactly one painted face is $6(n-2)^2$. Thus, $(n-2)^3 = 6(n-2)^2 \longrightarrow n-2 = 6 \longrightarrow n = 8$.

Grade Level 10 - NMT 2018 **Solutions**

1. **2** $\frac{1}{1-\frac{1}{x}} = x \longrightarrow 1 = x\left(1-\frac{1}{x}\right) \longrightarrow 1 = x-1 \longrightarrow x = 2.$

2. **9** The largest set of positive integers whose least common multiple is 100 is the set of <u>all</u> of the positive factors of 100. Note that if $N = p^r \cdot q^s$ where p and q are prime, N has $(r+1)(s+1)$ factors. Since $100 = 2^2 \cdot 5^2$, 100 has $(2+1)(2+1) = 9$ factors. They are: 1, 2, 4, 5, 10, 20, 25, 50, 100.

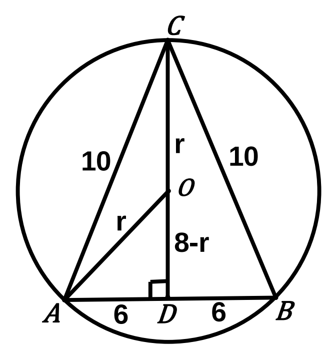
3. **15** If n is the number of sides of the polygon, then $\frac{360}{n}$ and $180 - \frac{360}{n}$ represent the measures of each of the exterior and interior angles, respectively. So, $180 - \frac{360}{n} - \frac{360}{n} = 132 \longrightarrow \frac{720}{n} = 48 \longrightarrow n = 15$. [Note: Taking the difference in the other order, $\frac{360}{n} - \left(180 - \frac{360}{n}\right) = 132$, does not yield an integer value of n.]

4. **1** Since there are 4 different cards, any card can be the third card from the left. The probability that card "A" is that third card is $\frac{1}{4}$. Thus $a^b = 1^4 = 1$.

5. **8** Since $\sqrt[4]{9} = \sqrt{3}$, $\sqrt{3} - \sqrt{\frac{1}{3}} = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$. Thus, $2+3+3 = 8$.

6. **40** In a triangle, the length of the segment joining the midpoints of two sides of the triangle is half the length of the third side. So, the perimeter of the second triangle is $\frac{1}{2}(40+50+70) = 80$, and the perimeter of the third triangle is $\frac{1}{2}(80) = 40$.

7. **625** If isosceles ΔABC is inscribed in circle $O, OA = OC = r$ and altitude $\overline{CD}$ passes through O and bisects $\overline{AB}$ so that $AD = DB = 6$. By the Pythagorean triple 6-8-10 in ΔACD, $CD = 8$, so $OD = 8 - r$. By the Pythagorean Theorem in ΔOAD, $6^2 + (8-r)^2 = r^2 \longrightarrow$ $36 + 64 - 16r + r^2 = r^2 \longrightarrow 16r = 100 \longrightarrow r = \frac{25}{4}$.
So, $100r = 100\left(\frac{25}{4}\right) = 625$.
Alternatively, the area of $\Delta ABC = \frac{1}{2}(12)(8) = 48$. If a triangle with sides of length $a, b,$ and c is inscribed in a circle of radius r, its area is $\frac{abc}{4r}$.
So, $\frac{10\cdot 10\cdot 12}{4r} = 48 \longrightarrow \frac{300}{r} = 48 \longrightarrow r = \frac{25}{4}$. Thus, $100r = 625$.



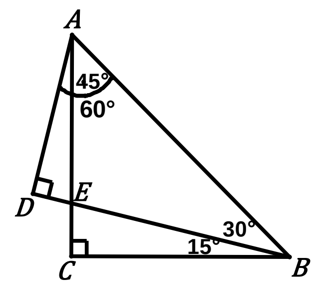
8. **36** The sum of the cubes of the roots, $p^3 + q^3 = (p+q)(p^2 - pq + q^2) =$ $(p+q)(p^2 + 2pq + q^2 - 3pq) = (p+q)[(p+q)^2 - 3pq] = 6[6^2 - 3(10)] = 36$.
Alternatively, $(p+q)^3 = p^3 + 3p^2q + 3pq^2 + q^3 \longrightarrow p^3 + q^3 = (p+q)^3 - 3pq(p+q) =$ $6^3 - 3(10)(6) = 36$.

9. **5** Let the perimeter of the equilateral triangle and the hexagon be P. By connecting the midpoints of the sides of the equilateral triangle, 4 smaller equilateral triangles are formed each of whose sides has length $\frac{P}{6}$. In the hexagon, by drawing segments from the center of the hexagon to each of its vertices, 6 equilateral triangles are formed each of whose sides also has length $\frac{P}{6}$. Since all of these smaller equilateral triangles are congruent and thus have equal

areas, the ratio of the area of the given equilateral triangle to the area of the hexagon is $4:6 = 2:3$. Thus, $2+3=5$.
Alternatively, let the side lengths of the given equilateral triangle and hexagon be $2x$ and x, respectively, so that each perimeter is $6x$. The area of the triangle is $\frac{(2x)^2\sqrt{3}}{4} = x^2\sqrt{3}$. The area of the hexagon is $6\left[\frac{x^2\sqrt{3}}{4}\right] = \frac{3x^2\sqrt{3}}{2}$. Thus, the required ratio is $\frac{x^2\sqrt{3}}{\frac{3x^2\sqrt{3}}{2}} = \frac{2}{3}$ and $2+3=5$.

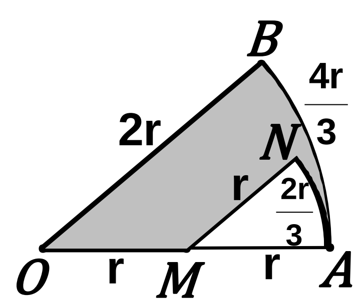
10. **37** Since ΔABC is obtuse, we know that:
Case 1: $AB+BC>AC$ and $(AB)^2+(BC)^2<(AC)^2$. So, $5+7>AC$ and $5^2+7^2<(AC)^2$. This yields $\sqrt{74}<AC<12$, so $AC=\{9,10,11\}$.
Case 2: $AB+AC>BC$ and $(AB)^2+(AC)^2<(BC)^2$. So, $5+AC>7$ and $5^2+(AC)^2<7^2$. This yields $2<AC<\sqrt{24}$, so $AC=\{3,4\}$. Thus, $AC=\{3,4,9,10,11\}$ and the required sum is 37.

11. **3** Since $\measuredangle D \cong \measuredangle C$ and $m\measuredangle DAE = m\measuredangle CBE = 15°$, $\Delta ADE \sim \Delta BCE$ by AA. In $30°$-$60°$-$90°$ ΔADB, $\frac{AD}{AB} = \frac{1}{2}$ and in $45°$-$45°$-$90°$ ΔACB, $\frac{AB}{BC} = \sqrt{2}$.
So, $\frac{AD}{AB} \cdot \frac{AB}{BC} = \frac{AD}{BC} = \frac{\sqrt{2}}{2}$. Thus, $\frac{[ADE]}{[BCE]} = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} = \frac{1}{2}$ and $1+2=3$.
(Note: The symbol $[ADE]$ means the area of ΔADE.)



12. **3** $f(4c) = 2\big(f(2c)\big)^2 - 1 \longrightarrow -\frac{1}{2} = 2\big(f(2c)\big)^2 - 1 \longrightarrow \big(f(2c)\big)^2 = \frac{1}{4} \longrightarrow f(2c) = -\frac{1}{2}$. So, $f(2c) = 2\big(f(c)\big)^2 - 1 \longrightarrow -\frac{1}{2} = 2\big(f(c)\big)^2 - 1 \longrightarrow \big(f(c)\big)^2 = \frac{1}{4} \longrightarrow f(c) = -\frac{1}{2}$. Thus, $-\frac{m}{n} = -\frac{1}{2} \longrightarrow \frac{m}{n} = \frac{1}{2}$ and $1+2=3$.

13. **9** Let $OB=OA=2r$, so $OM=MA=MN=r$. The length of arc $AB = \frac{\frac{2}{3}}{2\pi} \cdot 2\pi(2r) = \frac{4r}{3}$, and the length of arc $AN = \frac{\frac{2}{3}}{2\pi} \cdot 2\pi r = \frac{2r}{3}$. So, the perimeter of $ABOMN$ is $\frac{4r}{3} + 2r + r + r + \frac{2r}{3} = 18 \longrightarrow 6r = 18 \longrightarrow r = 3$. $\frac{[\text{sector } AOB]}{36\pi} = \frac{\frac{2}{3}}{2\pi} \longrightarrow [\text{sector } AOB] = 12$. Also, $\frac{[\text{sector } AMN]}{9\pi} = \frac{\frac{2}{3}}{2\pi} \longrightarrow [\text{sector } AMN] = 3$. Thus, the required area is $12-3=9$.



14. **96** Since each student sits next to his or her group member, each 2-member group counts as one unit. So, there are $(4-1)! = 6$ ways to arrange 4 groups in a circle. Within each group there are 2 seating arrangements, so the total number of seating arrangements is
$6 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 96$.

15.**16**

A pattern emerges if we set up a table:

Input	Sequences	# of operations C that yield output 1
1	1	0
2	2→1	1
3	3→2→1	2
4	4→2→1	2
5	5→3→2→1	3
6	6→3→2→1	3
7	7→4→2→1	3
8	8→4→2→1	3
9	9→5→3→2→1	4
…		4
16	16→8→4→2→1	4

One input needs 1 operation to output 1.
Two inputs need 2 operations to output 1.
Four inputs need 3 operations to output 1.
Eight inputs need 4 operations to output 1.
So, there are 2^{n-1} inputted numbers that require n operations C to output 1, where $n > 0$. So, there are 2^{5-1} numbers that require 5 operations C to output 1. Thus, $2^4 = 16$.

Grade Level 11 - NMT 2018 **Solutions**

1. **0** Lily's salary for year 2 was double her initial salary. Her salary for year 3 was one half of the year 2 salary. Therefore, her year 3 salary was equal to her year 1 salary. This is a net increase of 0 dollars.

2. **15** The radicand must be greater than or equal to zero. Therefore, $2x^2 + x - 105 \leq 0 \rightarrow (2x+15)(x-7) \leq 0 \rightarrow -\frac{15}{2} \leq x \leq 7$. In the interval $\left[-\frac{15}{2}, 7\right]$, there are 15 integers.

3. **15** Based on the fact that on the days they write, each writes for ten hours per day: Euler writes 30 pages per day every third day. Newton writes 10 pages per day every day. Gauss writes 20 pages per day on two days out of three. So, in the first three days, Euler writes 30 pages, Newton writes 30 pages, and Gauss writes 40 pages. Together, they write 100 pages in three days. So, working at the same rate, it takes them 15 days to write 500 pages.

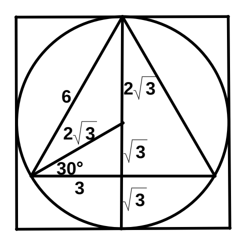
4. **3** We start by direct substitution: $f(-2) = -32a + 8b - 5 = -5 \rightarrow -32a + 8b = 0$. Note that $f(2) = 32a - 8b + 3 = 0 + 3 = 3$.

5. **20** Note that $(x+y)^2 = x^2 + y^2 + 2xy = 350 + 50 = 400 \rightarrow x + y = \pm 20$. We must use only the positive answer, 20.

6. **42** If $AC = x$, then $AB = x + 2$ and $BC = x - 7$. Use the Pythagorean Theorem in ΔABC: $x^2 + (x-7)^2 = (x+2)^2 \rightarrow 2x^2 - 14x + 49 = x^2 + 4x + 4 \rightarrow x^2 - 18x + 45 = 0 \rightarrow (x-15)(x-3) = 0 \rightarrow x = 3(\text{reject})$ or 15. Since ΔABC is an isosceles right triangle, $AD = DC = \frac{15\sqrt{2}}{2}$. So, the perimeter of quadrilateral $ABCD$ is $\frac{15\sqrt{2}}{2} + \frac{15\sqrt{2}}{2} + 17 + 8 = 25 + 15\sqrt{2}$. The required sum is 42.

7. **80** Let $y = |x - 20|$. Then $5y^2 - 19y + 12 = 0 \rightarrow (5y-4)(y-3) = 0 \rightarrow y = \frac{4}{5}$ or $3 \rightarrow |x - 20| = \frac{4}{5}$ or $3 \rightarrow x - 20 = \pm\frac{4}{5}$ or $\pm 3 \rightarrow x = \frac{104}{5}, \frac{96}{5}, 23, 17$. The sum of these roots is 80.

8. **48** The length of a side of the square is twice the radius of the circle. Draw an altitude of the equilateral triangle and a radius to one of the vertices of the triangle to create a 30°- 60° right triangle. Then, the length of the radius is $2\sqrt{3}$, the length of a side of the square is $4\sqrt{3}$, and the area of the square is 48.



9. **6** Rewrite the given equation as $\log_{72} x^2 + \log_{72} y^3 = 1 \rightarrow \log_{72} x^2y^3 = 1 \rightarrow x^2y^3 = 72$. Note that $72 = 3^2 \cdot 2^3$. Therefore, since x and y are positive integers, $x = 2$ and $y = 3$. The required product is 6.

10. **2** Note that the only difference between the two given equations is the distinction between addition and subtraction. Often, the given ellipse is known as the "associated ellipse" of the given hyperbola. The vertices of the hyperbola $(-1,0)$ and $(3,0)$ are the endpoints of the minor axis of the ellipse. These are the only 2 points of intersection. Alternatively, solve the given equations for $\frac{(x-1)^2}{4}$ and set the results equal to get : $1 - \frac{y^2}{9} = 1 + \frac{y^2}{9} \rightarrow y = 0 \rightarrow \frac{(x-1)^2}{4} = 1 \rightarrow (x-1)^2 = 4 \rightarrow x - 1 = \pm 2 \rightarrow x = 3$ or -1. There are two points of intersection.

11. **5** Use the trigonometric identity $\sin(A-B)=\sin A\cos B-\cos A\sin B$ to get $6\sin\left(ax-\frac{3\pi}{2}\right)=6\sin(ax)\cos\frac{3\pi}{2}-6\cos(ax)\sin\frac{3\pi}{2}=6\cos(ax)=5$. Alternatively, use the fact that the graphs of $y=\cos x$ and $y=\sin\left(x-\frac{3\pi}{2}\right)$ coincide. Therefore, $6\sin\left(ax-\frac{3\pi}{2}\right)=6\cos(ax)=6\cdot\frac{5}{6}=5$.

12. **8** The given series is an infinite geometric series with a ratio of $\frac{1}{2^x}$. Its sum is $\frac{a_1}{1-r}=\frac{\frac{1}{2^x}}{1-\frac{1}{2^x}}=\frac{1}{255}\rightarrow\frac{255}{2^x}=1-\frac{1}{2^x}\rightarrow\frac{256}{2^x}=1\rightarrow 2^x=256\rightarrow x=8$.

13. **144** The coordinates of the point of intersection above the x-axis of the parabola, $y^2=36x$ and the vertical line $x=4$ are $(4,12)$. Therefore the largest radius of a circular cross-section formed by revolving the given region over the $x-$axis is 12. The area of that circular cross-section is 144π. Thus, $a=144$.

14. **36** Denote the length of a side of square $ABCF$ as x. Using the given fact that the area of trapezoid $ABCE$ is 120, we have $\frac{1}{2}x(x+8)=120\rightarrow x^2+8x=240\rightarrow x^2+8x-240=0\rightarrow(x+20)(x-12)=0\rightarrow x=12$. When, in right ΔDCE, an altitude is drawn to the hypotenuse, the length of the altitude is the mean proportional between the lengths of the segments formed on the hypotenuse: $12^2=4\cdot FD\rightarrow FD=36$.

15. **126** We can use the "stars and bars" method of solution here:
Let 10 stars represent the 10 pencils. Between the 10 stars are 9 spaces. We wish to separate the 10 pencils into 5 groups by placing 4 bars in the 9 spaces, one bar in each space. The diagram below shows the 10 stars separated into groups of 1, 1, 1, 1, and 6. The number of ways we can place these 4 bars in these 9 spaces is ${}_9C_4=\frac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2\cdot1}=126$.
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Alternatively, start by giving each student one pencil. Then we have 5 additional pencils to distribute among the five students. Separate the 5 students with 4 dividers and treat the students as spaces between the dividers. We compute the number of arrangements by inserting the 5 students in the spaces. So we are choosing 5 out of 9 (5 students and 4 dividers). ${}_9C_5=\frac{9!}{5!\cdot4!}=\frac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2}=126$.

Grade Level 12 - NMT 2018 **Solutions**

1. **20** $f'(x) = 2(x+5) - 2(x-5) = 2x + 10 - (2x - 10) = 20.$
Alternatively, $f(x) = x^2 + 10x + 25 - (x^2 - 10x + 25) = 20x \longrightarrow f'(x) = 20.$

2. **11** In the rectangle, $m + n = 132$ and $mn = 12$. Thus, $\frac{1}{m} + \frac{1}{n} = \frac{m+n}{mn} = \frac{132}{12} = 11.$

3. **601** From the quadratic formula, the distance between the x-intercepts of a quadratic function is given by $d = \frac{-b+\sqrt{b^2-4ac}}{2a} - \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{2\sqrt{b^2-4ac}}{2a} = \frac{\sqrt{b^2-4ac}}{a}$. So, $d = \frac{\sqrt{(-25)^2-4(1)(6)}}{1} = \sqrt{625-24} = \sqrt{601}$. Thus, $d^2 = 601.$

4. **415** If $\{a_n\}_{n=1}^{\infty} = 1, 4, 7, 10, 13, 16, 19, 22, 25, 28, \ldots$, then $\{a_{a_n}\} = a_1, a_4, a_7, a_{10}, \ldots = 1, 10, 19, 28, \ldots.$
This is another arithmetic sequence with an initial term of 1 and a common difference of 9.
This means that $a_{a_{10}} = 1 + 9(9) = 82$ and that $\sum_{n=1}^{10} a_{a_n} = \frac{10}{2}(1 + 82) = 5(83) = 415.$

5. **60** The faucet adds water at a rate of $\frac{1}{10}$ pools per hour, and the pump removes water at a rate of $\frac{1}{12}$ pools per hour. Running at the same time, the amount of water in the pool will increase at $\frac{1}{10} - \frac{1}{12} = \frac{1}{60}$ pools per hour. So, the pool will fill in 60 hours.

6. **289** By successive substitution, $f(2) = f(1+1) = 1 + 2 + 1 = 4$; $f(3) = f(2+1) = 4 + 4 + 1 = 9$; $f(4) = f(3+1) = 9 + 6 + 1 = 16$. Spotting the pattern, $f(x) = x^2$, so $f(17) = 289$.
[Note: the identity $(x+1)^2 = x^2 + 2x + 1$ is the equation on which the question is based.]

7. **418** If we add 2 coins to the pile, the number of coins will be divisible by 3, 4, 5, 6, and 7. Since the least common multiple of 3, 4, 5, 6, and 7 is 420, there are 418 coins in the pile.

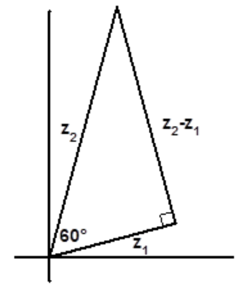
8. **60** There are $\frac{6!}{3!2!} = 5 \cdot 4 \cdot 3 = 60$ permutations of PEPPER, so $\frac{1}{p} = \frac{1}{60}$. Thus, $p = 60$.
Alternatively, deal out the cards one at a time and compute the probability of each card separately, based on which cards are left in the pile at each point: $\frac{1}{p} = \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{60}$.

9. **6** By the power of a point, $(AB)(AC) = (AD)^2$. Let $AB = BC = x$ and $AD = y$. So, $(x)(2x) = y^2 \longrightarrow 2x^2 = y^2 \longrightarrow x\sqrt{2} = y \longrightarrow 2x = y\sqrt{2}$, thus $AC = y\sqrt{2}$. The area of $\Delta ACD = \frac{1}{2}y\left(y\sqrt{2}\right)\sin 60° = \frac{1}{2}y^2\sqrt{2} \cdot \frac{\sqrt{3}}{2} = \frac{y^2\sqrt{6}}{4}$. If $y = \sqrt[4]{96}$, then $y^2 = \sqrt{96}$, so the area of $\Delta ACD = \frac{\sqrt{96}\sqrt{6}}{4} = \frac{4\sqrt{6}\sqrt{6}}{4} = 6.$

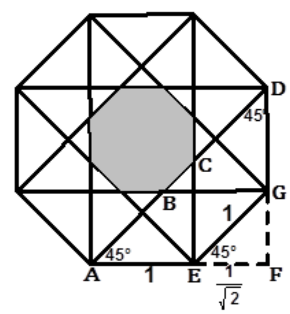
10. **88** Since the length of each side is 1, $P(n) = n$. By drawing n radii and n apothems, divide the polygon into $2n$ right triangles. Using right triangle trigonometry and the formula $A = \frac{1}{2}$(apothem)(perimeter), $A(n) = \frac{n}{4\tan\frac{\pi}{n}}$. So, $\lim_{n\to\infty} \frac{7n^2}{\frac{n}{4\tan\frac{\pi}{n}}} = \lim_{n\to\infty} \frac{28\tan\frac{\pi}{n}}{n^{-1}}$, which, by L'Hopital's Rule, equals $\lim_{n\to\infty} \frac{28\sec^2\left(\frac{\pi}{n}\right)(-\pi n^{-2})}{-n^{-2}}$. So, $\lim_{n\to\infty} 28\pi\sec^2\left(\frac{\pi}{n}\right) = 28\pi \approx 28\left(\frac{22}{7}\right) = 88$. [Note: If a closer approximation of π is used, the final answer is unaffected.]
Alternatively, as $n \longrightarrow \infty$, the regular n-gon approaches a circle. Let r be the radius of this circumscribed circle. So, $\lim_{n\to\infty} \frac{7[P(n)]^2}{A(n)} = \frac{7(2\pi r)^2}{\pi r^2} = 28\pi \approx 88.$

11. **91** The measure of the angle formed by the radii drawn to the two given points is $137° - 17° = 120°$. Since both radii (65 and 39) are divisible by 13, we can temporarily scale down the radii lengths by a factor of 13 to simplify computation using the points $(3, 17°)$ and $(5, 137°)$ instead. Let x represent the distance between these two points. Applying the Law of Cosines, $x^2 = 3^2 + 5^2 - 2(3)(5)\cos 120° = 9 + 25 - 30\left(-\frac{1}{2}\right) = 34 + 15 = 49$. Thus, $x = 7$ and scaling back up by a factor of 13 yields a distance of 91.

12. **26** Use the derivative of the given function $\frac{dy}{dx} = 3x^2 - 1$ at the point $(-1, 0)$ to find an equation of the tangent line: $y = 2x + 2$. Setting equations of the given function and the tangent line equal yields $x^3 - 3x - 2 = 0$. Since the line is tangent to the given function at the point where $x = -1$, $(x + 1)$ must be a factor of $x^3 - 3x - 2$. Divide to get $(x + 1)(x^2 - x - 2) = 0 \longrightarrow (x + 1)(x + 1)(x - 2) = 0 \longrightarrow x = 2$. Substitution in either function yields the other point of intersection $(2, 6)$. Thus, $10(2) + 6 = 26$.



13. **33** Since the complex number $z_2 = 2(\cos 60° + i \sin 60°)z_1$ has twice the magnitude of z_1 and is rotated $60°$ about the origin, the triangle formed by the vectors, z_1, z_2, and $z_2 - z_1$ contains a $60°$ angle with adjacent sides in a $2:1$ ratio. Therefore, this is a $30°$-$60°$-$90°$ triangle where $|z_2 - z_1|$ is the length of the side opposite the $60°$ angle. Thus, $|z_2 - z_1| = \sqrt{3}\sqrt{363} = \sqrt{3}\sqrt{3 \cdot 121} = 3 \cdot 11 = 33$.



14. **322** In a regular octagon, each interior angle is $135°$. The diagonals drawn from each vertex of this given octagon trisects its angles, so each angle is $45°$ and each triangle is an isosceles right triangle. Extend sides $\overline{AE}$ and $\overline{DG}$ to point F, forming isosceles right ΔAFD. It is given that $AE = EG = GD = 1$. Since $ABGE$ and $CDGE$ are parallelograms, $AB = CD = EG = 1$. In isosceles right ΔEFG, $EF = FG = \frac{1}{\sqrt{2}}$. The hypotenuse of ΔAFD, $AD = \sqrt{2}\,(AF) = \sqrt{2}\left(1 + \frac{1}{\sqrt{2}}\right) = \sqrt{2} + 1$. So, $AB + BC + CD = AD \longrightarrow 1 + BC + 1 = \sqrt{2} + 1 \longrightarrow BC = \sqrt{2} - 1$. Thus, $EG : BC = 1 : (\sqrt{2} - 1)$ and the ratio of the areas is $1^2 : (\sqrt{2} - 1)^2 = 1 : (3 - 2\sqrt{2})$. Thus, $100(3) + 10(2) + 2 = 322$.

15. **332** Let $\left(a, \frac{1}{a^2}\right)$ be the point of tangency. So, $y = \frac{1}{x^2} \longrightarrow \frac{dy}{dx} = -\frac{2}{x^3}\Big|_{x=a} = -\frac{2}{a^3}$. An equation of the tangent line is $y - \frac{1}{a^2} = -\frac{2}{a^3}(x - a)$. If $x = 0$, the y-intercept is $P\left(0, \frac{3}{a^2}\right)$. If $y = 0$, the x-intercept is $Q\left(\frac{3a}{2}, 0\right)$. The distance $PQ = \sqrt{\frac{9}{a^4} + \frac{9a^2}{4}}$, which is minimized when its radicand is minimized. Let radicand $R = 9a^{-4} + \frac{9}{4}a^2$, so $\frac{dR}{da} = -36a^{-5} + \frac{9}{2}a = 0 \longrightarrow -36 + \frac{9}{2}a^6 = 0 \longrightarrow a^6 = 8 \longrightarrow a^2 = 2$. Thus, $PQ = \sqrt{\frac{9}{4} + \frac{18}{4}} = \sqrt{\frac{27}{4}} = \frac{3\sqrt{3}}{2}$, and $100(3) + 10(3) + 2 = 332$.

1. **30** If we multiply the first equation by 3, we get the equivalent equation $6x + 15y = 30$. Thus, $k = 30$.

2. **999** Let $a = x + 10$. So, $a^2 - 1007a - 2018 = 0 \longrightarrow (a - 1009)(a + 2) = 0 \longrightarrow a = 1009 \longrightarrow x + 10 = 1009 \longrightarrow x = 999$.

3. **14** Let the other x-intercept be $(k, 0)$. Since the minimum point $(4, -3)$ is the parabola's vertex, its x-coordinate is the average of the parabola's x-intercepts. So, $\frac{-6+k}{2} = 4 \longrightarrow k = 14$.

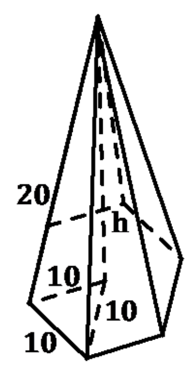
4. **409** The number $\left(\frac{20}{18}\right)^2 = 1.\overline{234567901}$. The block of numbers that repeat, called the repetend, has 9 digits. The sum of the first 99 digits is $11(2 + 3 + 4 + 5 + 6 + 7 + 9 + 0 + 1) = 407$. The 100^{th} digit is 2. So, $407 + 2 = 409$.

5. **80** Start with $f(3) = 2$. Then, $f(4) = 2f(3) = 4$; $f(5) = 2f(4) = 8$; $f(6) = f(5) + f(4) + 4 = 16$; $f(7) = f(6) + f(5) + 4 = 28$; $f(8) = f(7) + f(6) + 4 = 48$; and $f(9) = f(8) + f(7) + 4 = 48 + 28 + 4 = 80$.

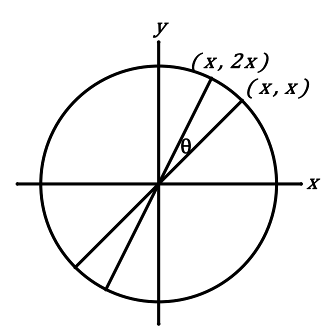
6. **30** Since points A and A' are the same point, $A(3, 5)$ must be the center of dilation. The scale factor is $\frac{A'B'}{AB} = 3$. A dilation of a line will not change its slope, so the image of $3x + 5y = 30$ is $3x + 5y = c$. To find c, we chose a point, $D\ (5, 3)$ on the given line and find its image, $D'(x, y)$, under the given dilation. Since $A'D' : AD = 3 : 1$, $x - 3 = 3(5 - 3) \longrightarrow x = 9$ and $y - 5 = 3(3 - 5) \longrightarrow y = -1$. Substituting $(9, -1)$ into $3x + 5y = c$ yields $c = 22$. Thus, $3 + 5 + 22 = 30$.

7. **502** Compute the number of factors of 2 in the given numeral by adding the integer parts of the following fractions: $\frac{2018}{2}$, $\frac{2018}{4}$, $\frac{2018}{8}$, $\frac{2018}{16}$, $\frac{2018}{32}$, $\frac{2018}{64}$, $\frac{2018}{128}$, $\frac{2018}{256}$, $\frac{2018}{512}$, and $\frac{2018}{1024}$. The sum of these integers is $1009 + 504 + 252 + 126 + 63 + 31 + 15 + 7 + 3 + 1 = 2011$. Every 4 factors of 2 gives a factor of 16, and thus a zero at the right of the hexadecimal representation of the given number. So, the integer part of $\frac{2011}{4}$ is 502.

8. **375** By drawing line segments from the center to each vertex, divide the hexagonal base into 6 equilateral triangles, each of whose sides is 10 cm. Consider the right triangle formed by the lateral edge, 20; the altitude of the pyramid, h,; and the base radius, 10. By the Pythagorean Theorem, $h^2 + 10^2 = 20^2 \longrightarrow h = 10\sqrt{3}$. The area of the pyramid's base is $6\left(\frac{10^2\sqrt{3}}{4}\right) = 150\sqrt{3}$. The volume of the pyramid is $V = \frac{1}{3}Bh = \frac{1}{3}\left(150\sqrt{3}\right)\left(10\sqrt{3}\right) = 1500\ \text{cm}^3$. Thus, the mass is $1500\ \text{cm}^3\left(\frac{0.25\text{g}}{\text{cm}^3}\right) = 375\ \text{g}$.



9. **325** Let θ be the measure of the smaller angle determined by the intersection of the two given lines. The line $y = x$ forms a $45°$ angle with the positive x-axis, and the line $y = 2x$ forms an angle measuring Arc tan $2 = 63.435°$ with the positive x-axis. So, $\theta =$



$63.435° - 45° = 18.435°$, and the area of the sector is $\frac{\theta}{360°}(\pi r^2) = \frac{18.435°}{360°}(2018\pi) \approx 325.$

10. **12** If the 3 students seated together count as one object, we place them at the 10-person table, fixing their position, and leaving 7 others to permute in $7!$ ways. Additionally, the 3 students can permute in 3! ways, so the total number of ways of arranging the 3 students seated together is $3! \cdot 7!$. If the students are not necessarily seated together, then the total number of ways of arranging 10 students at the table is $9!$ because we fix one student's position at the table and permute the 9 others in 9! ways. Thus, the required probability is $\frac{3! \cdot 7!}{9!} = \frac{1}{12}$ and $n = 12$.

Team Problem Solving - NMT 2018 **Solutions**

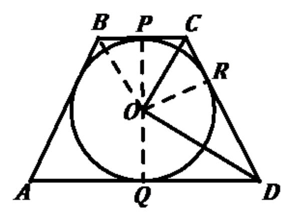
1. **107** Arrange the numbers in ascending order: 32, 33, 34, 35, 35, 36, 36, 36, and 47. The mean, the average of the numbers, is 36; the median, the middle number, is 35; and the mode, the most-frequent number, is 36. Thus, $36 + 35 + 36 = 107$.

2. **70** Factor 1120 into $2^5 \cdot 5 \cdot 7$. So, if $N = 2 \cdot 5 \cdot 7 = 70$, then $1120 \div N = 2^4 = 16$.

3. **35** The total amount of salt in the mixture is $6(.44) + 9.5(.30) = 5.49$. So, the salt percentage is $\frac{5.49}{15.5} = .3541 \ldots \approx 35\%$. Thus, to the nearest whole number, $n = 35$.

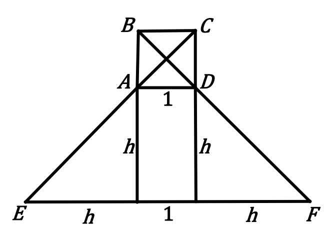
4. **54** For all integers m and n, Pythagorean triples can be generated using the equations: $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$. We want the perimeter and area to have the same numerical value. So, $a + b + c = \frac{ab}{2} \longrightarrow (m^2 - n^2) + 2mn + (m^2 + n^2) = \frac{2mn(m^2-n^2)}{2} \longrightarrow$ $2m^2 + 2mn = mn(m^2 - n^2) \longrightarrow 2m(m+n) = mn(m+n)(m-n) \longrightarrow m = n + \frac{2}{n}$. There are only two values of n that can be used: 1 and 2. When $n = 1$, $m = 3$ and the triple is 6-8-10. When $n = 2$, $m = 3$ and the triple is 5-12-13. Thus, the sum of their perimeters is $24 + 30 = 54$.
Alternatively, we can experiment with common Pythagorean triples, and with trial and error quickly conclude that 6-8-10 and 5-12-13 satisfy the given condition.

5. **96** The circle intersects the two bases of the trapezoid at their midpoints P and Q; so the area of trapezoid $ABCD$ is twice the area of trapezoid $PCDQ$. Let R be the point of tangency of $\overline{CD}$, with $\Delta OCR \cong \Delta OCP$ by hypotenuse-leg. Similarly, $\Delta ODR \cong \Delta ODQ$. It follows that the area of trapezoid $PCDQ$ is twice the area of ΔCOD, and therefore, the area of trapezoid $ABCD$ is four times the area of ΔCOD. So, $4 \cdot 24 = 96$.



6. **39** For all points in quadrant II, in piecewise form, the given function is: $y = -(x+8) + 2x + 48 = x + 40$, if $-40 < x < -8$, and $y = (x+8) + 2x + 48 = 3x + 56$, if $-8 \le x < 0$. Since both pieces of this function are linear with integral coefficients, for every integer value of x, there is an integer of value of y. So, if we consider only lattice points, the domain of the function is the set of all integers such that $-39 \le x \le -1$. Thus, there are 39 lattice points.

7. **20** Let h be the height of the trapezoid. The area of the trapezoid is $\frac{h}{2}(1 + 2h + 1) = 6 \longrightarrow h^2 + h - 6 = 0 \longrightarrow (h+3)(h-2) = 0 \longrightarrow$ $h = 2 \longrightarrow 10h = 20$.

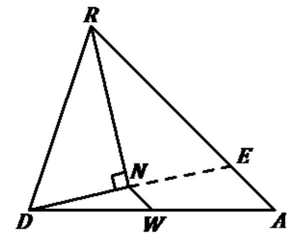


8. **41** First, consider the positive divisors of 12: 1, 2, 3, 4, 6, and 12. Their sum is 28. Combine the sum of their reciprocals into a single fraction: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{12+6+4+3+2+1}{12}$. Notice that the numerator is the sum of the divisors of 12 and the denominator is 12. The same idea can be used for the sum of the reciprocals of the divisors of 240. The resulting single fraction is $\frac{744}{240}$. This reduces to $\frac{31}{10}$ and the required sum is 41.

9. **5** Complete the squares for x and for y to get $(x-4)^2+(y+6)^2=49$. The center of the circle is $(4,-6)$ and the radius is 7. Thus, the required sum is $4-6+7=5$.

10. **110** Take the logarithm with base 10 of both sides and apply rules for logarithms to yield: $(\log x)^2 = 3\log x - 2$. Then let $a=\log x \longrightarrow a^2-3a+2=0 \longrightarrow (a-2)(a-1)=0 \longrightarrow \log x = 2$ or $\log x = 1 \longrightarrow x=100$ or $x=10$. Thus, the sum of these roots is 110.

11. **18** Extend $\overline{DN}$ through N until it meets $\overline{RA}$ and label the intersection E. Notice that $\Delta RND \cong \Delta RNE$ by ASA, so $NE=ND$ and $RE=RD=85$. Consequently, $AE=121-85=36$. Since N and W are midpoints of two sides of ΔDEA, $NW=\frac{1}{2}AE$, so $NW=18$.

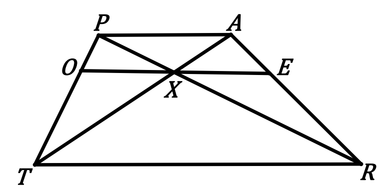


12. **15** Since factors always appear in pairs, the only integers that have an odd number of factors are perfect squares. Every integer has 1 and the number itself as factors. Therefore, we are looking for the third factor to be a prime. The perfect squares of primes less than 2500 are the squares of 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47. There are 15 such integers.

13. **27** Since $4\times P$ is a single digit, $P=1$ or 2. Since $4\times S$ results in P, P must be even, so $P=2$. It follows that $S=8$. Since $4\times A$ is less than 10, $A=0$ or 1. Since $4\times T+3$ must be odd, $A=1$ and $T=7$. Test the remaining digit to see that $R=9$, so $STRAP=87912$ and the sum of the digits is 27.

14. **49** By the change-of-base rule, $\frac{\log\sqrt{x}}{\log 3}\cdot\frac{\log 12}{\log 7}\cdot\frac{\log 81}{\log 12}=4 \longrightarrow \frac{\frac{1}{2}\log x}{\log 3}\cdot\frac{4\log 3}{\log 7}=4 \longrightarrow \frac{2\log x}{\log 7}=4 \longrightarrow \frac{\log x}{\log 7}=2 \longrightarrow \log_7 x = 2 \longrightarrow x=49$.

15. **20** Without loss of generality, let $AP=14$ and $TR=35$. Since $\Delta PAX \sim \Delta RTX$, $\frac{14}{35}=\frac{PX}{XR}=\frac{AX}{TX}$. Since $\Delta POX \sim \Delta PTR$, $\frac{OX}{35}=\frac{PX}{PR}=\frac{14}{49} \longrightarrow OX=10$. Since $\Delta AXE \sim \Delta ATR$, $\frac{XE}{35}=\frac{AX}{AT}=\frac{14}{49} \longrightarrow XE=10$. Therefore, $OE=OX+XE=20$.



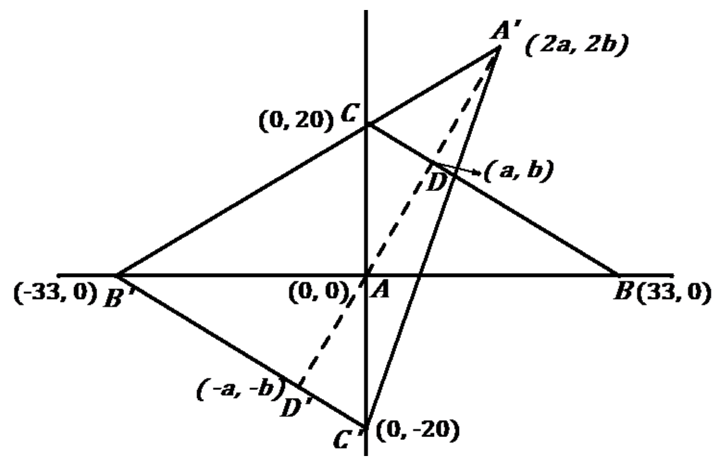
16. **483** The least common multiple, LCM(5, 6, 8) = 120. This means that 123 is the smallest number that leaves a remainder of 3 when divided by 5, 6, or 8. Since we need the result to be between 450 and 500, multiply 120 by 4 and add 3 to get 483.

17. **12** Multiply the given equation by $12(\sec x + \tan x)$ to get $12(\sec^2 x - \tan^2 x) = 24(\sec x + \tan x)$. Since $\sec^2 x - \tan^2 x = 1$, $24(\sec x + \tan x) = 12$.

18. **242** Let $x=-2018$ and $y=2018$. So, $f(-2018+2018) = -2018+f(2018) \longrightarrow f(0) = -2018+f(2018) \longrightarrow -1776 = -2018+f(2018) \longrightarrow f(2018)=242$.

19. **23** Make a list of all possible outcomes where the third number is the sum of the first two numbers: **(1, 1, 2), (1, 2, 3)**, (1, 3, 4), (1, 4, 5), (1, 5, 6), **(2, 1, 3), (2, 2, 4), (2, 3, 5), (2, 4, 6),** (3, 1, 4), **(3, 2, 5)**, (3, 3, 6), (4, 1, 5), **(4, 2, 6)**, (5, 1, 6). There are 15 possible cases and 8 contain at least one 2. The probability that a 2 was rolled is $\frac{8}{15}$ and the required sum is 23.
Alternatively: Consider the total number of possible outcomes. If the first die is 1, there are 5 possible values for the second die. If the first die is 2, there are only 4 possibilities for the second die. Continue this way until the first die is 5 when there is only one possible value for the second die. Therefore, there are $5+4+3+2+1=15$ cases in total. If the first die shows 2, there are 4 possible cases. If the second die shows 2, there are 4 cases, and if the third die shows 2, there is only 1 case. Since there is an overlap with (2, 2, 4), there are $4+4+1-1=$ 8 cases that have the number 2, so the probability is $\frac{8}{15}$ and the required sum is 23.

20. **990** From point A, draw $\overline{AD} \perp \overline{BC}$ with D on $\overline{BC}$, and let the coordinates of D be (a, b). If point A reflects over $\overleftrightarrow{BC}$, to A', the coordinates of A' are $(2a, 2b)$. The coordinates of B' and C' are $(-33, 0)$ and $(0, -20)$, respectively. If D reflects over A to point D', the coordinates of D' are $(-a, -b)$, and since $\overline{BC} \parallel \overline{B'C'}$, $\overline{AD'} \perp \overline{B'C'}$. So, points $A', D, A,$ and D' are collinear and $\overline{A'D'}$ is an altitude of $\Delta A'B'C'$. The equations of $\overleftrightarrow{BC}$ and $\overleftrightarrow{A'D'}$ are, respectively, $y=-\frac{20}{33}x+20$ and $y=\frac{33}{20}x$, and their point of intersection is point $D: (a, b) =$ $(8.865\ldots, 14.627\ldots)$. The length of $\overline{B'C'}$ is $\sqrt{33^2+20^2}=\sqrt{1489}$. The length of altitude $\overline{A'D'}$ is $\sqrt{(3a)^2+(3b)^2}$. Thus, the area of $\Delta A'B'C' =$ $\frac{1}{2}(\sqrt{1489})(\sqrt{(3a)^2+(3b)^2} = 990$.



Tie Breakers - NMT 2018 **Solutions**

1.

2.

3.