Nassau County Interscholastic Mathematics League

9

Grade 9

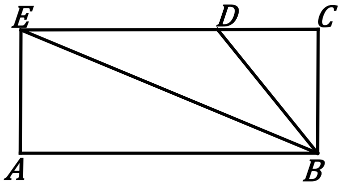
TEAM #

Mathematics Tournament 2017

No calculators may be used on this part.
All answers will be integers from 0 to 999 inclusive.
One (1) point for each correct answer.

Name ______________________ **School** ______________________ **Score** __________

Time Limit: 45 minutes

Lower Division	*Answer Column*
1. Compute the area of ΔABC whose vertices are $A(-2, 3)$, $B(-1, 7)$ and $C(8, -1)$.	1.
2. Compute the sum of three consecutive positive odd integers if the square of the second integer exceeds three times the sum of the largest and smallest integers by 135.	2.
3. If p and r are the roots of the equation $3x^2 - x - 10 = 0$, compute the value of the product $(p-2)(r-2)$.	3.
4. In the diagram, $\overline{BE}$ is a diagonal of both rectangle $ABCE$ and trapezoid $ABDE$. If $AE = 10, BE = 26$, and $ED = 16$, compute the area of ΔBCD. 	4.
5. When $16\sqrt{243}$ is divided by $3\sqrt{72}$, the quotient, in simplest radical form, is $a\sqrt{b}$. Compute $a+b$.	5.
6. The surface area of a sphere is $144\pi \text{ in}^2$. If the volume of this sphere is $k\pi \text{ in}^3$, compute k.	6.
7. Using the letters of the word "SQUARE," compute the number of 6-letter arrangements that can be made if each letter is used exactly once, if each arrangement begins with a vowel, and if the vowels and consonants alternate.	7.
8. When the radius of a right circular cylinder is increased by 25%, the volume of the cylinder is unchanged, and the height of the cylinder is decreased by k%. Compute k.	8.

Turn Over

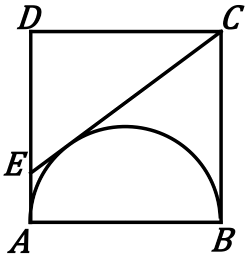
9

Mathematics Tournament 2017

Grade 9

Time Limit: 45 minutes

Lower Division

	Answer Column
9. The sum of two numbers, a and b, is 8, and their product is 3. Compute the value of $a^3 + b^3$.	9.
10. If A, B, and C represent three numbers such that $1001C - 2002A = 4004$ and $1001B + 3003A = 5005$, compute the average of A, B, and C.	10.
11. Given the equation $\frac{2016}{2017}x - 1 + \frac{1}{x} = 0$ with roots r_1 and r_2. If the sum of the reciprocals of r_1 and r_2, in simplest form, is p, compute p.	11.
12. Compute the units digit of 2017^{2017}.	12.
13. Bryanna and Sunil run in opposite directions on a circular track, starting at diametrically opposite points. They first meet after Bryanna has run 100 meters. The next time they meet, Sunil has run 150 meters past their first meeting point. If each one runs at a constant speed, compute, in meters, the length of the track.	13.
14. Samuel has 5 green cards numbered 1 through 5, and 4 yellow cards numbered 3 through 6. He stacks the cards so that the colors alternate and so that the number on each green card divides evenly into the number on each neighboring yellow card. Compute the sum of the numbers on the three middle cards.	14.
15. In the diagram, each side of square $ABCD$ has a length of 4. A semicircle with diameter $\overline{AB}$ is constructed inside the square, and the tangent to the semicircle from C intersects side $\overline{AD}$ at E. Compute the length of $\overline{CE}$. 	15.

10

Nassau County Interscholastic Mathematics League

Grade 10

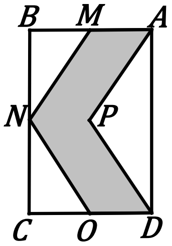
TEAM # ☐

Mathematics Tournament 2017

No calculators may be used on this part.
All answers will be integers from 0 to 999 inclusive.
One (1) point for each correct answer.

Name ______________________ **School** ______________________ **Score** __________

Time Limit: 45 minutes **Lower Division** *Answer Column*

1. In the figure of rectangle $ABCD$, $AB = 12$ and $BC = 18$, where points M, N, and O are midpoints of the sides $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ respectively, and point P is the center of the rectangle. Compute the area of the shaded region $AMNODP$. 	1.
2. The least common multiple of the positive integers $3, m$, and n is 27. The greatest common divisor of the same three positive integers is 3. Compute the minimum possible value of $3mn$.	2.
3. Compute the length of the smaller dimension of a rectangle whose perimeter is 74 and whose area is 340.	3.
4. Compute a positive integer that has exactly 8 factors if two of them are 15 and 35.	4.
5. Each edge of a pyramid whose base is a square has a length of 4. If its volume, in simplest form, can be expressed as $\frac{a\sqrt{b}}{c}$, compute $a + b + c$.	5.
6. Three numbers form an arithmetic sequence with a common difference of 12. If the second and the third numbers are interchanged, the new numbers form a geometric sequence. Compute the sum of the absolute values of the numbers.	6.
7. A regular hexagon and an equilateral triangle are inscribed in a circle the length of whose radius is 2. If the area of the equilateral triangle is E, and the area of the regular hexagon is H, compute the product, EH.	7.
8. If $f(x+1) = ax^2 + bx + c$, $f(0) = 201$, and $f(2) = 7$, compute $a + c$.	8.

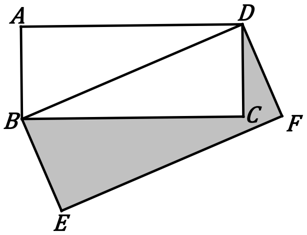
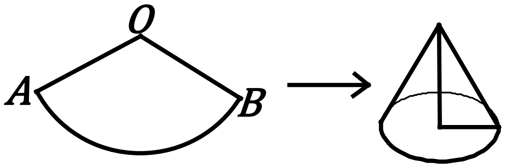
Turn Over

10

Mathematics Tournament 2017

Grade 10

Time Limit: 45 minutes **Lower Division**

	Answer Column
9. In the figure, rectangle $ABCD$ and rectangle $BEFD$ are similar, where $\overline{BD}$ is a diagonal of rectangle $ABCD$. If $AB = 5$ and $BC = 12$, then the area of the shaded region $BEFDC$ is expressible in simplest form as $\frac{p}{q}$. Compute $p + q$. 	9.
10. As shown in the diagram, sector AOB is used to create a right circular cone. The lengths of the radius and height of the cone are 5 and 12, respectively. If the perimeter of the sector is $a + b\pi$, compute $a + b$. 	10.
11. The endpoints of $\overline{AB}$ have coordinates $A(3, 0)$ and $B(-1, -3)$. The image of $\overline{AB}$ after a rotation about point Q is $\overline{A'B'}$ whose endpoints are $A'(-1, -8)$ and $B'(2, -12)$. If the coordinates of point Q are (x, y), compute $x^2 + y^2$.	11.
12. If the roots of $(c-2)^3 + (c-8)^3 = (2c-10)^3$ are x, y, and z, where $x < y < z$, compute $x^3 + y^2 + z$.	12.
13. If the infinite continued fraction equation, $x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}}$ is expressed as $x = a + \sqrt{b}$, where $\sqrt{b}$ is in simplest radical form, compute $a + b$.	13.
14. When the sum $4^{17} + 6^{17}$ is divided by 25, compute the remainder.	14.
15. From 4 boys and 4 girls, choose 4 teams of 2 students per team. The probability, in simplest form, of choosing exactly 2 teams each with 1 boy and 1 girl is $\frac{p}{q}$. Compute $p + q$.	15.

Nassau County Interscholastic Mathematics League

11

Grade 11

TEAM #

Mathematics Tournament 2017

No calculators may be used on this part.
All answers will be integers from 0 to 999 inclusive.
One (1) point for each correct answer.

Name ______________________ **School** ______________________ **Score** __________

Time Limit: 45 minutes **Upper Division** *Answer Column*

1. If $2^x = 8^{y+2}$ and $27^{2x} = 9^y$, compute the value of $\lvert x + y \rvert$.	1.
2. If ΔABC is inscribed in circle O, the measure of arc BC is 120°, $AB = 12$, $AC = 8$, and the area of ΔABC is expressed in simplest form as $p\sqrt{q}$. Compute the product pq.	2.
3. Compute the middle term of $(1 + i)^8$.	3.
4. Compute $\sum_{k=1}^{4} \cos^2\left(\frac{k\pi}{4}\right)$.	4.
5. For a given parabola $y = f(x)$, the coordinates of the y-intercept are (0,6) and an equation of the axis of symmetry is $x = a$. Compute $3f(2a) - 5$.	5.
6. Compute the length of a radius of the circle that contains points whose coordinates are (0,0), (8,0) and (0,6).	6.
7. Compute the number of integral solutions of the inequality $6x^2 - 63 < 13x$.	7.
8. Three terms of a geometric sequence are $x - 5$, $x^2 - 25$, and $3x^3 - 15x^2 - 75x + 375$. Compute the positive value of x.	8.

Turn Over

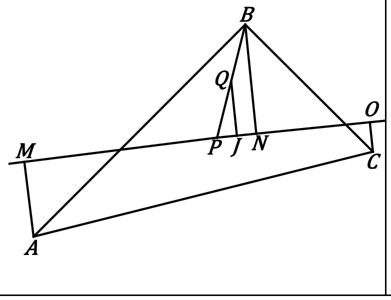
11

Mathematics Tournament 2017

Grade 11

Upper Division

Time Limit: 45 minutes

	Answer Column
9. If $\log_{64} x + \log_4 x + \log_8 x = 5$, compute x.	9.
10. Compute the number of possible domains for a function if each domain must be a subset of $\{1,2,3,4,5,6,7,8\}$.	10.
11. If $\cos x - \sin x = \frac{4}{5}$ and $\cos 2x = \frac{3}{5}$, compute the value of $12(\cos x + \sin x)$.	11.
12. Compute the units digit of $\frac{3^{33}+5^{12}}{3^{11}+5^{4}}$.	12.
13. The school copy room has two copy machines, both of which are used to complete one job. Working alone, machine A can complete the job in 3 hours, and machine B can complete the same job in 5 hours. Both machines start working at the same time and each machine works at a constant rate. While both machines are working on the same job, machine A stops for 45 minutes and machine B continues to work. Let $\frac{p}{q}$, in simplest form, represent the time it takes for the job to be completed. Compute $p + q$.	13.
14. If $f\left(\frac{x-1}{2}\right) = \frac{14-2x}{x^2-2x+37}$ and $f(x) = -\frac{2}{29}$, compute the integral value of x.	14.
15. A line is drawn through point P, the centroid of ΔABC. On the drawn line, points M, N and O are the feet of perpendiculars from vertices A, B, and C, respectively. Point Q is the midpoint of $\overline{BP}$ and point J is the foot of the perpendicular from point Q to $\overleftrightarrow{MO}$. If $QJ = 12$, compute the sum $AM + BN + CO$. 	15.

12

Nassau County Interscholastic Mathematics League

Grade 12

TEAM # ☐

Mathematics Tournament 2017

No calculators may be used on this part.
All answers will be integers from 0 to 999 inclusive.
One (1) point for each correct answer.

Name ____________________ **School** ____________________ **Score** __________

Time Limit: 45 minutes **Upper Division** *Answer Column*

<table>
<tr><td>1. Compute $2^{3^2} - (2^3)^2$.</td><td>1.</td></tr>
<tr><td>2. If $f(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$, compute $f\,'(-4)$.</td><td>2.</td></tr>
<tr><td>3. For complex z, $f(z) = iz + 8$, where $i = \sqrt{-1}$. Compute $f((2 + 2i)^2)$.</td><td>3.</td></tr>
<tr><td>4. In regular hexagon $ABCDEF$, compute the measure of the acute angle, in degrees, formed by the intersection of the diagonals $\overline{AC}$ and $\overline{BF}$.</td><td>4.</td></tr>
<tr><td>5. If $\log(x + 24) + \log(x - 24) = 2$, compute x.</td><td>5.</td></tr>
<tr><td>6. Compute the number of six-digit numbers $(100000 \leq x < 1000000)$ whose digits are a permutation of the digits of 614209. Note: the number 614209 itself *does* count as a valid permutation.</td><td>6.</td></tr>
<tr><td>7. In ΔABC, $m\measuredangle A = 15°$, $m\measuredangle C = 90°$, and $AB = 16$. Compute the area of ΔABC.</td><td>7.</td></tr>
<tr><td>8. Given the values shown in the table. If $g(x) = f\left(f\left(f(x)\right)\right)$, compute $g'(1)$.
<table>
<tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>$f(x)$</td><td>2</td><td>4</td><td>6</td><td>8</td></tr>
<tr><td>$f'(x)$</td><td>3</td><td>5</td><td>7</td><td>9</td></tr>
</table>
</td><td>8.</td></tr>
</table>

Turn Over

12

Mathematics Tournament 2017

Grade 12

Time Limit: 45 minutes

Upper Division

	Answer Column
9. If $x_1 = 2$ $x_1 + x_2 = 3$ $x_1 + x_2 + x_3 = 5$ ... $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 17$, such that $\sum_{k=1}^{n} x_k$ is the nth prime number, compute $10x_1 + 9x_2 + 8x_3 + 7x_4 + 6x_5 + 5x_6 + 4x_7$.	9.
10. Compute the distance between the points whose polar coordinates are $(3, 30°)$ and $(5, 150°)$.	10.
11. In a sequence of numbers, $a_{n+1} = \sum_{k=1}^{n} a_k$ for all $n \geq 1$. If $a_{2014} = 117$, compute a_{2017} .	11.
12. Ilene is playing a game. She rolls a fair six-sided die. If she rolls a 6, she wins. If she rolls anything else, she rolls again. If her two rolls now add up to exactly 6, she wins. If she fails to roll a 6 on one roll or a total of 6 on two rolls, she loses. The probability that Ilene wins, in lowest terms is $\frac{p}{q}$. Compute $p + q$.	12.
13. Let $f(x)$ be a function which has the property that, for all real a and b, $f(a + b) = f(a) + f(b)$. If $f(369) = 123$, compute $f(246)$.	13.
14. Consider the cube bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x = 6$, $y = 6$, and $z = 6$. The planes $x + y + z = 6$ and $x + y + z = 12$ cut the cube into three solids. Compute the volume of the largest of these three solids.	14.
15. A line, drawn tangent to the ellipse $5x^2 + 12y^2 = 83$ at the first-quadrant point (a, b), passes through the point $(4.15, 0)$. Compute $360ab$.	15.

M

Nassau County Interscholastic Mathematics League

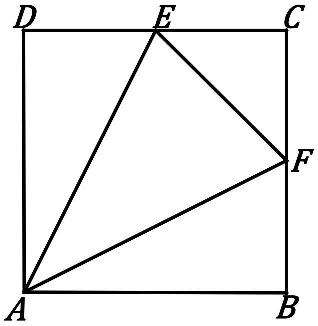
Mathletics

TEAM #

Mathematics Tournament 2017

Calculators may be used on this part.
All answers will be integers from 0 to 999 inclusive.
One (1) point for each correct answer.

Name ____________________ **School**____________________ **Score** ________

Time Limit: 30 minutes	*Answer Column*
1. The function $g(x)$ has the property that $g(x) = 1 - g(x-1)$ for all positive integral values of x. If $g(1) = 314$, compute $g(2017)$.	1.
2. The positive root of the equation $(10x-3)^2 - (4x-9)^2 = 0$ is $\frac{a}{b}$, where a and b are positive integers that are relatively prime. Compute $a+b$.	2.
3. In square $ABCD$, $AB = 40$, and points E and F are the midpoints of sides $\overline{DC}$ and $\overline{BC}$, respectively. Compute the area of ΔAEF. 	3.
4. Adam has a circular swimming pool with an 18-foot diameter and a constant 4-foot depth. The bottom of the pool and the surface of the pool have the same area. Water begins to leak out of the pool from the bottom at a constant rate of 2 gallons/hour. If no water is added, compute the <u>least</u> number of complete days necessary for all of the water to drain from the pool. (note: 1 cubic foot = 7.614 gallons)	4.
5. Compute the number of positive integer divisors of 20^{17}.	5.

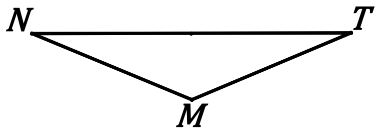
Turn Over

M

Mathematics Tournament 2017

Mathletics

Time Limit: 30 minutes

	Answer Column
6. Each term of the sequence $\{S_n\}$ is defined as the sum of n consecutive odd integers starting with 7. The first few terms are: $S_1 = 7$, $S_2 = 7 + 9 = 16$, and $S_3 = 7 + 9 + 11 = 27$. Compute the greatest integer n satisfying $S_n < 2017$.	6.
7. Compute the distance between the lines $24x + 7y = 31$ and $24x + 7y = 1331$.	7.
8. The circle $x^2 + y^2 = 2100$ contains the points A and B, each of which has an x-coordinate of 30. Lines l and m are tangent to the circle at points A and B, respectively, and intersect at point P. Compute the x-coordinate of point P.	8.
9. Twelve standard 6-sided dice are rolled. Let p represent the probability that exactly two dice land on each different numbered face. That is, exactly two dice land on the face numbered 1, two dice land on the face numbered 2, … , and two dice land on the face numbered 6. Compute $10{,}000p$ rounded to the nearest integer.	9.
10. Isosceles ΔNMT with $NT = 24$ and $NM = MT = 13$ is rotated about $\overleftrightarrow{NT}$ to form a 3-dimensional solid. The volume of this solid is $k\pi$, where k is an integer. Compute k. 	10.

T

Nassau County Interscholastic Mathematics League

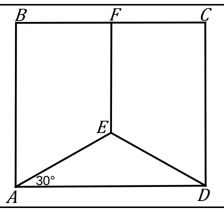
Team Problem Solving

TEAM #

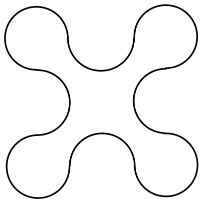
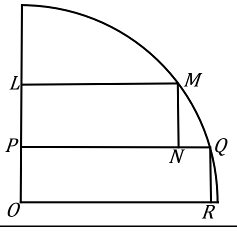
Mathematics Tournament 2017

HAND IN ONLY **ONE** ANSWER SHEET PER TEAM
Calculators may be used on this part.
All answers will be integers from 0 to 999 inclusive.
Three (3) points per correct answer.

Team Copy **School** ____________________ **Score** __________

Time Limit: 60 minutes	*Answer Column*
1. Define $a\ @\ b\ @\ c$ as the mean of the the three numbers, a, b, and c. If c is a two-digit number, compute the greatest possible integer value of $34\ @\ 16\ @\ c$.	1.
2. If $\frac{p+q}{p-q} = \frac{279}{261}$, compute $\frac{p}{q}$.	2.
3. The coordinates of the midpoints of the sides of ΔABC are $(-2, 5), (3, 4)$, and $(1, 1)$. If the coordinates of the vertices of ΔABC are $A(d, e)$, $B(f, g)$, and $C(h, k)$, compute $d + e + f + g + h + k$.	3.
4. Compute the value of b so that the parabola, defined by $f(x) = 5x^2 - bx + 45$, is tangent to the positive x-axis.	4.
5. If $\log_b(x) = 237$, compute the value of $\log_{\frac{1}{b}}\left(\frac{1}{x}\right)$.	5.
6. If $20^{x+y} \cdot 25^{x-2y} = 4^4 \cdot 10^{30}$, compute $x^2 - y^2$.	6.
7. In rectangle $ABCD$, point E is equidistant from A, D, and $\overline{BC}$. If $m\measuredangle EAD = 30°$, and the area of ΔEAD is 100 square units, compute the number of square units in the area of rectangle $ABCD$. 	7.
8. The coordinates of the endpoints of a line segment are $(21, 63)$ and $(909, 618)$. Compute the number of points on the line segment that have integer values for both the x- and y-coordinates. [Note: these points are called lattice points.]	8.
9. The endpoints of a diagonal of a rhombus have coordinates $(0, 10)$ and $(10, 0)$. If the coordinates of the endpoints of the other diagonal are (p, q) and (r, s), compute the sum $p + q + r + s$.	9.
10. The number 1881 is the sum of 19 consecutive integers. Compute the sum of the largest and smallest of these consecutive integers.	10.

Turn Over

11. Compute the greatest possible quotient when the five-digit number $1abcd$, whose missing digits are unique prime numbers, is divided by 495.	11.
12. If x and y are positive integers, compute the number of solutions, (x, y), for the equation $x^2 - y^4 = 3 \cdot 7 \cdot 29$.	12.
13. The cube of the sum of two integers exceeds the sum of the cubes of the two integers by 1404. If the sum of the two integers is 13, compute the product of the two integers.	13.
14. The quantity x varies directly as the sum of the two quantities y and z. The quantity y varies directly as the difference $z - x$. When $x = 20$, $y = -15$ and $z = 25$. Compute $(xy)^2$ when $z = 5$.	14.
15. Each arc in the diagram is either a quarter-circle with radius 1 or a semi-circle with radius 1. The area of the shape can be written in the form $a + b\pi$. Compute $a + b$. 	15.
16. If $\frac{1}{5} = \frac{1}{a} + \frac{1}{b}$, where a and b are different positive integers, compute ab.	16.
17. Compute the least integer greater than 100 that has exactly 6 positive integer divisors.	17.
18. Compute the value of n that satisfies the equation $2^n = 4^{12} + 8^8 + 16^6 + 64^4$.	18.
19. In the quarter-circle shown, rectangle $OPQR$ has $OP = 7$ and $OR = 24$. If, in rectangle $PLMN$, the ratio $\frac{LM}{LP} = \frac{5}{2}$, compute the area of rectangle $PLMN$. 	19.
20. There are three values of x for which the expressions $x^2 + 3x - 1$ and $2x^2 - 3x + 6$ will be pairs of consecutive positive odd integers. Compute the sum of the three pairs of consecutive odd integers.	20.

T

Nassau County Interscholastic Mathematics League

Team Problem Solving

TEAM #

Mathematics Tournament 2017

DO NOT HAND IN THIS COPY. HAND IN THE ONE TEAM COPY.
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All answers will be integers from 0 to 999 inclusive.
Three (3) points per correct answer.

Individual Copy

Time Limit: 60 minutes

	Answer Column
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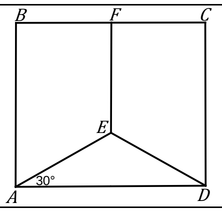
4. Compute the value of b so that the parabola, defined by $f(x) = 5x^2 - bx + 45$, is tangent to the positive x-axis.	4.
5. If $\log_b(x) = 237$, compute the value of $\log_{\frac{1}{b}}\left(\frac{1}{x}\right)$.	5.
6. If $20^{x+y} \cdot 25^{x-2y} = 4^4 \cdot 10^{30}$, compute $x^2 - y^2$.	6.
7. In rectangle $ABCD$, point E is equidistant from A, D, and $\overline{BC}$. If $m\sphericalangle EAD = 30°$, and the area of ΔEAD is 100 square units, compute the number of square units in the area of rectangle $ABCD$. 	7.
8. The coordinates of the endpoints of a line segment are $(21, 63)$and $(909, 618)$. Compute the number of points on the line segment that have integer values for both the x- and y-coordinates. [Note: these points are called lattice points.]	8.
9. The endpoints of a diagonal of a rhombus have coordinates $(0, 10)$ and $(10, 0)$. If the coordinates of the endpoints of the other diagonal are (p, q) and (r, s), compute the sum $p + q + r + s$.	9.
10. The number 1881 is the sum of 19 consecutive integers. Compute the sum of the largest and smallest of these consecutive integers.	10.

Turn Over

T

Mathematics Tournament 2017

Team Problems

Time Limit: 60 minutes	*Answer Column*
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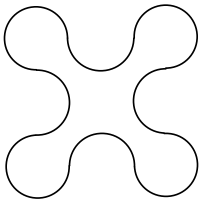
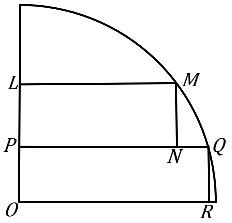
12. If x and y are positive integers, compute the number of solutions, (x, y), for the equation $x^2 - y^4 = 3 \cdot 7 \cdot 29$.	12.
13. The cube of the sum of two integers exceeds the sum of the cubes of the two integers by 1404. If the sum of the two integers is 13, compute the product of the two integers.	13.
14. The quantity x varies directly as the sum of the two quantities y and z. The quantity y varies directly as the difference $z - x$. When $x = 20$, $y = -15$ and $z = 25$. Compute $(xy)^2$ when $z = 5$.	14.
15. Each arc in the diagram is either a quarter-circle with radius 1 or a semi-circle with radius 1. The area of the shape can be written in the form $a + b\pi$. Compute $a + b$. 	15.
16. If $\frac{1}{5} = \frac{1}{a} + \frac{1}{b}$, where a and b are different positive integers, compute ab.	16.
17. Compute the least integer greater than 100 that has exactly 6 positive integer divisors.	17.
18. Compute the value of n that satisfies the equation $2^n = 4^{12} + 8^8 + 16^6 + 64^4$.	18.
19. In the quarter-circle shown, rectangle $OPQR$ has $OP = 7$ and $OR = 24$. If, in rectangle $PLMN$, the ratio $\frac{LM}{LP} = \frac{5}{2}$, compute the area of rectangle $PLMN$. 	19.
20. There are three values of x for which the expressions $x^2 + 3x - 1$ and $2x^2 - 3x + 6$ will be pairs of consecutive positive odd integers. Compute the sum of the three pairs of consecutive odd integers.	20.

Grade Level 9 - NMT 2017 **Solutions**

1. **22** Circumscribe a rectangle about ΔABC. In square units, the area of the rectangle is $10 \cdot 8 = 80$, and the areas of the three right triangles surrounding ΔABC are 36, 20, and 2. So, the area of $\Delta ABC =$ (the area of the rectangle)$-$ (the sum of the areas of the three surrounding triangles). Thus, area of $\Delta ABC = 80 - (36 + 20 + 2) = 22$.

2. **45** Let $x - 2$, x, and $x + 2$ represent the three consecutive positive odd integers. So, $x^2 = 3(2x) + 135 \longrightarrow x^2 - 6x - 135 = 0 \longrightarrow (x - 15)(x + 9) = 0 \longrightarrow x = 15$ or $x = -9$. Thus, the required integers are 13, 15, 17, and their sum is 45.

3. **0** $(3x + 5)(x - 2) = 0 \longrightarrow x = -\frac{5}{3}$ or $x - 2$. Thus, $\left(-\frac{5}{3} - 2\right)(2 - 2) = 0$. Alternate solution: The sum and product of the roots of the given equation are $\frac{1}{3}$ and $-\frac{10}{3}$, respectively. Thus, $(p - 2)(r - 2) = pr - 2(p + r) + 4 = -\frac{10}{3} - 2\left(\frac{1}{3}\right) + 4 = 0$.

4. **40** In right ΔAEB, by the Pythagorean Theorem, $AB = \sqrt{26^2 - 10^2} = 24 = EC$. Since $ED = 16$, then $DC = 8$, and $AE = BC = 10$. Thus, the area of $\Delta BCD = \frac{1}{2} \cdot 8 \cdot 10 = 40$.

5. **10** $\frac{16\sqrt{243}}{3\sqrt{72}} = \frac{16\sqrt{81}\sqrt{3}}{3\sqrt{36}\sqrt{2}} = \frac{144\sqrt{3}}{18\sqrt{2}} = \frac{8\sqrt{3}}{\sqrt{2}} = \frac{8\sqrt{6}}{2} = 4\sqrt{6}$. Thus, the required sum is $4 + 6 = 10$.

6. **288** $A = 4\pi r^2 = 144\pi \longrightarrow r^2 = 36 \longrightarrow r = 6$. So, $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(6)^3 = 288\pi$. Thus, $k = 288$.

7. **36** The desired arrangement is of the form $V_1C_1V_2C_2V_3C_3$, where V_k and C_k are vowels and consonants, respectively. The number of ways of choosing the three vowels is $3 \cdot 2 \cdot 1$, and the number of ways of choosing the three consonants is also $3 \cdot 2 \cdot 1$. Thus, the total number of 6-letter arrangements is $3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 36$.

8. **36** Let r and h_1 be the radius and height of the first cylinder, and let $1.25r$ and h_2 be the radius and height of the second cylinder. Since the cylinders have equal volume, $\pi r^2 h_1 = \pi(1.25r)^2 h_2 \longrightarrow h_1 = (1.25)^2 h_2 \longrightarrow h_2 = 0.64h_1 = (1 - 0.36)h_1$. Thus, the height of the cylinder was decreased by 36%, and so $k = 36$.

9. **440** Since $a + b = 8$ and $ab = 3$, $(a + b)^2 = a^2 + 2(3) + b^2 = 64 \longrightarrow a^2 + b^2 = 58$. So, $a^3 + b^3 = (a + b)(a^2 - ab + b^2) = (8)(58 - 3) = 440$. Alternate solution: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a + b) = a^3 + b^3 + 9 \cdot 8 = 512$. Thus, the required sum is $512 - 72 = 440$.

10. **3** Add the two given equations to get $1001A + 1001B + 1001C = 9009$. Thus, $A + B + C = 9$ and the average, $\frac{A+B+C}{3} = 3$.

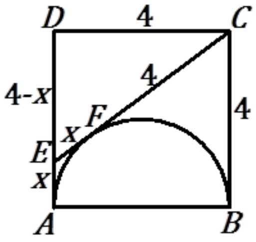
11. **1** Rewriting the given equation in standard quadratic form yields $2016x^2 - 2017x + 2017 = 0$. The sum of the roots, $r_1 + r_2 = \frac{2017}{2016}$. The product of the roots, $r_1r_2 = \frac{2017}{2016}$. The sum of the reciprocals of the roots, $\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1+r_2}{r_1r_2}$. Thus, $\frac{2017}{2016} \Big/ \frac{2017}{2016} = 1$.

12. **7** The first several powers of the digit 7 are: $7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$, $7^5 = 16807, \ldots$. We see that that the units digits of the powers of 7, and therefore the powers of 2017, repeat in a cycle of 4: 7, 9, 3, 1, 7, 9, 3, 1,… Since the exponent 2017 has a remainder of 1 when divided by 4, the units digit of 2017^{2017} has the same units digit as 2017^1, or 7.

13. **350** Let Bryanna start at point B and Sunil start at point S, where $\overline{BS}$ is a diameter of the circular track. Let their first meeting point be A, with arc BA having a length of 100 meters, and with the lengths of arcs BA and AS having a sum equal to half the circumference of the circle. Since each person travels at a constant speed, then when they meet for the second time, they have traveled an additional complete circle. So, since Bryanna travels twice as far between point A and the second meeting point as she did from point B to point A, she covered an additional 200 (twice her original distance) meters. So, the circle's circumference is the sum of her 200 meters and Sunil's 150 meters, the distance they travel between point A and the second meeting point. Thus, the length of the track is 350 meters.

14. **12** Let the green cards be G1,G2,G3,G4,G5, and the yellow cards be Y3,Y4,Y5,Y6. G4 and G5 divide evenly into only Y4 and Y5. Therefore the stack must begin G4,Y4,… and end Y5,G5 or the reverse. G2 divides evenly into only Y4 and Y6; therefore, the stack starts with G4,Y4,G2,Y6 and ends Y5,G5 or the reverse. G3 divides evenly into Y3 and Y6; therefore, the stack becomes G4,Y4,G2,Y6,G3,Y3,G1,Y5,G5 or the reverse. Either way, the sum of the three middle cards, $6 + 3 + 3 = 12$.

15. **5** If tangent $\overline{CE}$ intersects the semicircle at point F, then, as shown in the diagram, $AE = FE = x$, and $CF = CB = 4$ because tangent segments drawn to a circle from an external point are congruent. Using the Pythagorean Theorem in ΔDEC, $4^2 + (4 - x)^2 = (4 + x)^2 \longrightarrow$ $x^2 - 8x + 32 = x^2 + 8x + 16 \longrightarrow 16x = 16 \longrightarrow x = 1$. Thus, $CE = 4 + 1 = 5$.



Grade Level 10 - NMT 2017 **Solutions**

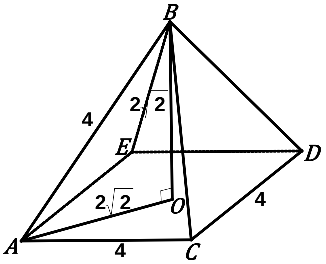
1. **108** Draw $\overline{NP}$ to separate the shaded region into two congruent parallelograms with common base $NP = \frac{1}{2}AB = 6$. The height of each parallelogram is $\frac{1}{2}BC = 9$. So, the area of each parallelogram is $6 \cdot 9 = 54$. Double that to obtain the required area of 108.

2. **243** Since the least common multiple of $3, m$, and n is 27, none of these numbers can be greater than 27. Since the greatest common divisor of $3, m$, and n is 3, none of these numbers can be less than 3. So the numbers are 3, 3, and 27 and their product is 243.

3. **17** Denote the dimensions of the rectangle by x and $37 - x$. Therefore, $x(37 - x) = 340 \rightarrow x^2 - 37x + 340 = 0 \rightarrow (x - 17)(x - 20) = 0 \rightarrow x = 17$ or 20. The smaller dimension of the rectangle is 17.

4. **105** Since $15 = 3 \cdot 5$ and $35 = 5 \cdot 7$, seven of the factors are 1, 3, 5, 7, 15, 21, and 35. The remaining factor must be $3 \cdot 5 \cdot 7 = 105$.

5. **37** Call the pyramid B-$ACDE$. To calculate its height, use ΔBAD with $BA = BD = 4$ and $AD = 4\sqrt{2}$ (the length of $\overline{AD}$, the diagonal of the square $ACDE$). Let point O be the center of the square base. Then, $AO = 2\sqrt{2}$ and ΔABO is an isosceles right triangle. So, $BO = 2\sqrt{2}$. The volume of the pyramid is $\frac{1}{3}Bh = \frac{1}{3} \cdot 4^2 \cdot 2\sqrt{2} = \frac{32\sqrt{2}}{3}$. The required sum is $32 + 2 + 3 = 37$



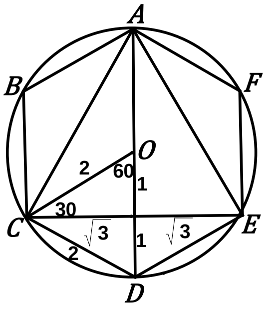
6. **28** The original arithmetic sequence is $a - 12,\ a,\ a + 12$. The geometric sequence is $a - 12,\ a + 12,\ a$. So, by the definition of a geometric sequence, $(a + 12)^2 = a(a - 12) \rightarrow a^2 + 24a + 144 = a^2 - 12a \rightarrow 36a = -144 \rightarrow a = -4$. The three original numbers are $-16, -4$, and 8. The required sum is $16 + 4 + 8 = 28$.

7. **54** Point O is the center of the circle, the center of equilateral ΔACE, and the center of regular hexagon $ABCDEF$.

The area of the equilateral triangle $= E = \frac{s^2\sqrt{3}}{4} = \frac{(2\sqrt{3})^2\sqrt{3}}{4} = 3\sqrt{3}$.

The area of the regular hexagon = $H = 6[\text{area } \Delta COD] = 6\left[\frac{2^2\sqrt{3}}{4}\right] = 6\sqrt{3}.$

Thus, the required product, $EH = 3\sqrt{3} \cdot 6\sqrt{3} = 54$.



8. **104** From the given function, if $x = -1$, $f(0) = a - b + c = 201$, and if $x = 1$, $f(2) = a + b + c = 7$. Add these equations to get $2(a + c) = 208 \rightarrow a + c = 104$.

9. **497** Since the corresponding sides of the similar rectangles are in proportion, $\frac{AB}{BE} = \frac{BC}{EF} \rightarrow \frac{AB}{BE} = \frac{BC}{BD} \rightarrow \frac{5}{BE} = \frac{12}{13} \rightarrow BE = \frac{65}{12}$. The area of the shaded region equals the area of rectangle $BDFE$ minus the area of ΔBDC. The area of the shaded region equals $\frac{65}{12} \cdot 13 - \frac{1}{2} \cdot 12 \cdot 5 = \frac{845}{12} - 30 = \frac{485}{12}$. The required sum is $485 + 12 = 497$.

10. **36** The length of a radius of the sector is the length of the slant height of the cone. By the Pythagorean Theorem, the slant height is $\sqrt{5^2+12^2}=13$. The length of the arc of the sector is the circumference of the cone's base. So, the length of the arc is $2\pi(5)=10\pi$. Thus, the perimeter of the sector is $13+13+10\pi=26+10\pi$, and the required sum is $26+10=36$.

11. **61** The center of rotation, Q is equidistant from points A and A', and from points B and B'. Thus, point Q is the intersection of the perpendicular bisectors of $\overline{AA'}$ and $\overline{BB'}$. The midpoint and slope of $\overline{AA'}$ are $(1,-4)$ and 2, respectively, and the midpoint and slope of $\overline{BB'}$ are $\left(\frac{1}{2},-\frac{15}{2}\right)$ and -3, respectively. Since perpendicular lines have negative reciprocal slopes, the equations of the perpendicular bisectors are $y+4=-\frac{1}{2}(x-1)$ and $y+\frac{15}{2}=\frac{1}{3}\left(x-\frac{1}{2}\right)$. Solving this system to find the point of intersection yields $-\frac{1}{2}x+\frac{1}{2}-4=\frac{1}{3}x-\frac{1}{6}-\frac{15}{2}\longrightarrow$ $-3x+3-24=2x-1-45\longrightarrow -3x-21=2x-46\longrightarrow 5x=25\longrightarrow x=5$ and $y=-6$. Thus, $5^2+(-6)^2=61$.

12. **41** Note that $(c-2)+(c-8)=2c-10$ and that whenever $a^3+b^3=(a+b)^3$, then $a^3+b^3=a^3+3a^2b+3ab^2+b^3\rightarrow 3a^2b+3ab^2=0\rightarrow 3ab(a+b)=0$.
So, if $a=c-2$ and $b=c-8$, then $3(c-2)(c-8)(2c-10)=0\rightarrow c=2,\ 8$, or $5\rightarrow$ $x=2,\ y=5,\ z=8$ and the required sum is $8+25+8=\ 41$.

13. **1** Assuming that the infinite continued fraction converges to a limit, then we can rewrite the given equation as $x=\frac{1}{2+x}\longrightarrow x^2+2x-1=0$. By the Quadratic Formula, the positive value of $x=\frac{-2+\sqrt{8}}{2}=\frac{-2+2\sqrt{2}}{2}=-1+\sqrt{2}$, and the required sum is $-1+2=1$.

14. **20** By the binomial expansion we have:
$4^{17}=(5-1)^{17}=5^{17}-{}_{17}C_1(5^{16})+{}_{17}C_2(5^{15})-\ \ldots\ -{}_{17}C_{15}(5^2)+{}_{17}C_{16}(5^1)-1$ and
$6^{17}=(5+1)^{17}=5^{17}+{}_{17}C_1(5^{16})+{}_{17}C_2(5^{15})+\ldots\ +{}_{17}C_{15}(5^2)+{}_{17}C_{16}(5^1)+1$.
So, $4^{17}+6^{17}=2(5^{17})+2\left({}_{17}C_2(5^{15})\right)+\ \ldots\ +2\left({}_{17}C_{14}(5^3)\right)+2\left({}_{17}C_{16}(5^1)\right)$.
Every term of $4^{17}+6^{17}$ is divisible by 5^2 except the last term. Thus, the required remainder is the remainder when $2\left({}_{17}C_{16}(5^1)\right)$ is divided by 25. So, $2\left({}_{17}C_{16}(5^1)\right)=2\cdot 17\cdot 5=170=25\cdot 6+20$. Thus, the remainder is 20.

15. **59** There are $\frac{{}_8C_2\cdot{}_6C_2\cdot{}_4C_2\cdot{}_2C_2}{4!}=105$ ways to choose 4 teams of 2 students per team (*). Since each of the two chosen teams must have exactly 1 boy and 1 girl, there are ${}_4C_2\cdot{}_4C_2\cdot 2!=72$ ways to do that (**). The required probability is $\frac{72}{105}=\frac{24}{35}$. The required sum is 59.
(*): To see why 4! is required, try two easier problems. First, separate 4 people into 2 groups of 2 and notice that division by 2 or 2! is required. Then, separate 6 people into 3 groups of 2 and see that division by 6 or 3! is required.
(**): ${}_4C_2$ is the number of ways of choosing 2 girls out of the 4 girls and also the number of ways of choosing 2 boys out of the 4 boys. 2! is required because, for each boy, there are 2 possible girls.

Grade Level 11 - NMT 2017 Solutions

1. **3** Re-write the given equations with common bases as $2^x = 2^{3y+6}$ and $3^{6x} = 3^{2y}$. Set the exponents equal and solve: $x = -\frac{3}{4}$ and $y = -\frac{9}{4} \rightarrow |x + y| = 3$.

2. **72** If the measure of arc BC is 120°, then the measure of inscribed $\measuredangle A$ is 60°. So, the area of $\Delta ABC = \frac{1}{2} bc \sin A = \frac{1}{2} \cdot 8 \cdot 12 \sin 60° = 24\sqrt{3}$. Thus, the required product is 72.

3. **70** Use the binomial expansion theorem to get the middle term: ${}_8C_4 \cdot 1^4 \cdot i^4 = 70 \cdot 1 \cdot 1 = 70$.

4. **2** Re-write the given sum as $\cos^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{2}\right) + \cos^2\left(\frac{3\pi}{4}\right) + \cos^2(\pi) = \left(\frac{\sqrt{2}}{2}\right)^2 + 0 + \left(-\frac{\sqrt{2}}{2}\right)^2 + (-1)^2 = \frac{1}{2} + 0 + \frac{1}{2} = 1 = 2$.

5. **13** Since $x = a$ is a line of vertical symmetry and $f(0) = 6$, then $f(2a) = 6$ also. Therefore, $3f(2a) - 5 = 13$. Alternatively, in vertex form, an equation of the parabola is $f(x) = (x - a)^2 + 6 - a^2$. So, $f(2a) = (2a - a)^2 + 6 - a^2 = 6$. Thus, $3f(2a) - 5 = 13$.

6. **5** The given points on the circle determine a right triangle with side lengths 6, 8, and 10. The circle is circumscribed about the right triangle. Hence, its diameter is the hypotenuse of the right triangle, 10, and its radius is 5.

7. **7** Re-write the inequality as $6x^2 - 13x - 63 < 0 \rightarrow (3x + 7)(2x - 9) < 0$. The solution interval is $-\frac{7}{3} < x < \frac{9}{2}$ which includes the integers $-2, -1, 0, 1, 2, 3, 4$. Thus, there are 7 integers on the solution interval.

8. **10** The geometric ratio is $\frac{x^2-25}{x-5} = \frac{3x^3-15x^2-75x+375}{x^2-25} \rightarrow \frac{(x+5)(x-5)}{x-5} = \frac{3x^2(x-5)-75(x-5)}{(x+5)(x-5)} \rightarrow$
$x + 5 = \frac{3x^2-75}{x+5} \longrightarrow (x + 5)^2 = 3x^2 - 75 \rightarrow x^2 + 10x + 25 = 3x^2 - 75 \rightarrow$
$2x^2 - 10x - 100 = 0 \rightarrow x^2 - 5x - 50 = 0 \rightarrow (x - 10)(x + 5) = 0 \rightarrow x = 10$ or -5.
The positive value of x is 10.

9. **32** Use the change of base formula to re-write the given equation as $\frac{\log x}{\log 64} + \frac{\log x}{\log 4} + \frac{\log x}{\log 8} = 5$.
Re-write this equation with the common denominator $\log 64$ or $6 \log 2$ as
$\frac{\log x}{6 \log 2} + \frac{3 \log x}{6 \log 2} + \frac{2 \log x}{6 \log 2} = 5 \rightarrow 6 \log x = 30 \log 2 \rightarrow \log x = 5 \log 2 \rightarrow x = 32$.

10. **256** There are 8 integer values in the domain interval. Each value can be included in the domain or omitted. Since there are 2 choices for each value, there are a total of 2^8 or 256 possible subsets. Alternatively, we are choosing values from 8 elements, so using combinations: $\sum_{k=0}^{8} {}_8C_k = 256$.

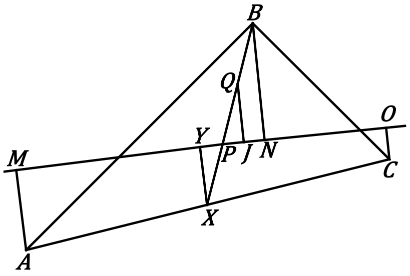
11. **9** $\cos 2x = \cos^2 x - \sin^2 x = (\cos x - \sin x)(\cos x + \sin x) \longrightarrow \frac{3}{5} = \frac{4}{5}(\cos x + \sin x)$.
Multiply both sides by 15 to get $12(\cos x + \sin x) = 9$.

12. **9** Re-write the given expression by factoring the numerator as the sum of two cubes: $\frac{(3^{11}+5^4)(3^{22}-3^{11}\cdot 5^4+5^8)}{3^{11}+5^4} = 3^{22} - 3^{11}\cdot 5^4 + 5^8$. Since $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243, \ldots$ the units digits of the powers of 3 repeat in a cycle of 4: 3, 9, 7, 1, … So, 3^{22} has the same units digit as 3^2 or 9, and 3^{11} has the same units digit as 3^3 or 7. Also, powers of 5 always have a units digit of 5. Since only the units digit is required, we use modular arithmetic:
$3^{22} - 3^{11}\cdot 5^4 + 5^8 \ (\mathrm{mod}10) \equiv 9 - 7\cdot 5 + 5(\mathrm{mod}10) \equiv -21 \equiv 9(\mathrm{mod}10)$.
So, the units digit is 9.

13. **107** Let x represent the number of hours that machine B works to complete the job, and let $x - \frac{3}{4}$ represent the number of hours that machine A works to complete the job. Working alone, machine A does $\frac{1}{3}$ of the job per hour, and machine B does $\frac{1}{5}$ of the job per hour. So, $\frac{x-\frac{3}{4}}{3} + \frac{x}{5} = 1 \longrightarrow 5x - \frac{15}{4} + 3x = 15 \longrightarrow x = \frac{75}{32}$, which is machine B's working time and the total time to complete the job. Thus, $75 + 32 = 107$.

14. **7** Let $a = \frac{x-1}{2} \rightarrow x = 2a + 1 \rightarrow f(x) = f(2a+1) = \frac{14-2(2a+1)}{(2a+1)^2-2(2a+1)+37} = \frac{12-4a}{4a^2+36} = \frac{3-a}{a^2+9} = -\frac{2}{29} \rightarrow$ $87 - 29a = -2a^2 - 18 \rightarrow 2a^2 - 29a + 105 = 0 \rightarrow (2a-15)(a-7) = 0$. The only integral solution is 7.

15. **48** $\Delta QPJ \sim \Delta BPN$ and from the given information, the corresponding sides are in a ratio of 1:2. Thus, $BN = 24$. Extend $\overrightarrow{BQP}$ until it intersects $\overline{AC}$ at point X. Let point Y be the foot of the perpendicular from point X to $\overline{MO}$. Since the centroid of ΔABC separates median $\overline{BX}$ in a 2:1 ratio, $QP = PX$. Also, $\Delta QPJ \cong \Delta XPY \rightarrow QJ = XY = 12$. Since point X is a midpoint of $\overline{AC}$, and $\overline{XY} \parallel \overline{AM} \parallel \overline{CO}$, $\overline{XY}$ is a median of trapezoid $AMOC$. Thus, $XY = \frac{1}{2}(AM + OC) = QJ = 12$. Therefore, $AM + OC = 24$ and the required sum $AM + BN + CO = 24 + 24 = 48$.



Grade Level 12 - NMT 2017 Solutions

1. **448** Since exponentiation is performed from top to bottom, $2^{3^2} - (2^3)^2 = 2^9 - 2^6 = 2^6(2^3 - 1) = 64(7) = 448$.

2. **405** Note that $f(x) = (x+1)^5$. So, $f\,'(x) = 5(x+1)^4$. Thus, $f'(-4) = 5(-4+1)^4 = 5(-3)^4 = 5(81) = 405$. Alternatively, we can differentiate term-by-term to get $f'(x) = 5x^4 + 20x^3 + 30x^2 + 20x + 5$. Thus, $f'(-4) = 1280 - 1280 + 480 - 80 + 5 = 405$.

3. **0** $(2+2i)^2 = 4 + 8i + 4i^2 = 4 + 8i - 4 = 8i$. So, $f(8i) = i(8i) + 8 = 8i^2 + 8 = -8 + 8 = 0$.

4. **60** Each interior angle of the regular hexagon measures $\frac{180(6-2)}{6} = 120°$. Let diagonals $\overline{AC}$ and $\overline{BF}$ intersect at point G. Since ΔABF and ΔBAC are isosceles triangles with vertex $\measuredangle FAB$ and vertex $\measuredangle CBA$ each measuring $120°$, $m\measuredangle ABF = m\measuredangle BAC = 30°$. Since $\measuredangle BGC$ (the acute angle between the diagonals) is exterior to ΔABG, its measure is the sum of the measures of its remote interior angles, $\measuredangle BAC$ and $\measuredangle ABF$. Thus, in degrees, $30 + 30 = 60$.

5. **26** Using the product rule of logarithms and the 10-24-26 Pythagorean triple, $\log(x+24) + \log(x-24) = 2 \longrightarrow \log(x^2 - 24^2) = 2 \longrightarrow x^2 - 24^2 = 10^2 \longrightarrow x^2 = 10^2 + 24^2 = 26^2$. Thus, $x = 26$, only. (Note: we reject $x = -26$ because it is not in the domain of the equation.)

6. **600** We can use any permutation of the digits, as long as the permutation does not start with 0. So, there are 5 choices for the first digit, 5 choices remaining for the second digit, 4 choices for the third, 3 for the fourth, 2 for the fifth, and 1 for the last digit. Thus, the number of permutations is $5 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 600$.

7. **32** In right ΔABC, using SOHCAHTOA, $AC = 16\cos 15°$ and $BC = 16\sin 15°$. Thus, the area of the triangle is $\frac{1}{2}(16\sin 15°)(16\cos 15°) = 128\sin 15°\cos 15° = 64 \cdot 2\sin 15°\cos 15° = 64\sin 30° = 64\left(\frac{1}{2}\right) = 32$. Alternatively, reflect ΔABC about side $\overline{AC}$ to create an isosceles triangle with legs of length 16 and a vertex angle with a measure of $30°$. Using the formula, $A = \frac{1}{2}ab\sin C$, the area of $\Delta ABC = \frac{1}{2}\left(\frac{1}{2} \cdot 16 \cdot 16 \cdot \sin 30°\right) = 32$.

8. **135** Using the chain rule twice: $g'(x) = f'\left(f\left(f(x)\right)\right) \cdot f'\left(f(x)\right) \cdot f'(x)$. So, $g'(1) = f'\left(f\left(f(1)\right)\right) \cdot f'\left(f(1)\right) \cdot f'(1) = f'\left(f(2)\right) \cdot f'(2) \cdot 3 = f'(4) \cdot 5 \cdot 3 = 9 \cdot 5 \cdot 3 = 135$.

9. **109** Each equation can be subtracted from its successor to obtain the individual x_i value. That is, $x_2 = 3 - 2 = 1$, $x_3 = 5 - 3 = 2$, $x_4 = 2$, $x_5 = 4$, $x_6 = 2$, and $x_7 = 4$. Thus, $10(2) + 9(1) + 8(2) + 7(2) + 6(4) + 5(2) + 4(4) = 109$. Alternatively, adding the seven given equations yields $7x_1 + 6x_2 + 5x_3 + 4x_4 + 3x_5 + 2x_6 + x_7 = 2 + 3 + 5 + 7 + 11 + 13 + 17 = 58$. Since $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 17$, we can triple this equation and add it to the previous equation to get $10x_1 + 9x_2 + 8x_3 + 7x_4 + 6x_5 + 5x_6 + 4x_7 = 58 + 3(17) = 109$.

10. **7** The two given points and the origin determine a triangle with two sides of 3 and 5 and an included angle of $120°$. If the distance between the two given points is d, then by the Law of Cosines, $d^2 = 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cdot \cos 120° = 9 + 25 - 30\left(-\frac{1}{2}\right) = 49$. Thus, $d = 7$.

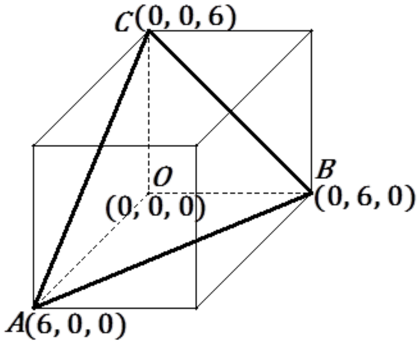
Alternatively, convert both points to rectangular form and use the distance formula: $(3, 30°) \longrightarrow (3\cos 30°, 3\sin 30°) = \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ and $(5, 150°) \longrightarrow (5\cos 150°, 5\sin 150°) = \left(\frac{-5\sqrt{3}}{2}, \frac{5}{2}\right)$. So, $d = \sqrt{\left(\frac{3\sqrt{3}}{2} - \frac{-5\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2} - \frac{5}{2}\right)^2} = \sqrt{\left(4\sqrt{3}\right)^2 + (-1)^2} = \sqrt{48 + 1} = 7.$

11\. **936** By writing the given sum in expanded form, we see that $a_{n+1} = (a_1 + a_2 + a_3 + \cdots + a_{n-1}) + a_n$, and that $a_n = a_1 + a_2 + a_3 + \cdots + a_{n-1}$. So, $a_{n+1} = (a_n) + a_n = 2a_n$. Thus, the sequence is a geometric sequence with a common ratio 2. Therefore, $a_{2017} = a_{2014} \cdot 2 \cdot 2 \cdot 2 = 117 \cdot 8 = 936$.

12\. **47** The probability that Ilene wins in one roll is $\frac{1}{6}$. In order to win in two rolls, she needs to roll something other than a 6 on the first roll. Regardless of which number from 1 through 5 was rolled, there is exactly one roll which will yield a sum of 6. So, there is a $\frac{5}{6} \cdot \frac{1}{6} = \frac{5}{36}$ chance that she wins on the second roll. Therefore, the probability that Ilene wins is $\frac{1}{6} + \frac{5}{36} = \frac{11}{36}$ and the required sum is $11 + 36 = 47$.

13\. **82** Since $f\big((a+b)+c\big) = f(a+b) + f(c) = f(a) + f(b) + f(c)$, then $f(369) = f(123) + f(123) + f(123)$. So, $3f(123) = 123 \longrightarrow f(123) = 41$. Thus, $f(246) = f(123) + f(123) = 41 + 41 = 82$.

14\. **144** Since $A(6,0,0)$, $B(0,6,0)$, and $C(0,0,6)$ are vertices of the cube and lie on the plane $x + y + z = 6$, the section of the cube cut by the plane nearest the origin is the triangular pyramid C-OAB. The base of the pyramid, ΔAOB has area $\frac{1}{2} \cdot OA \cdot OB = \frac{1}{2} \cdot 6 \cdot 6 = 18$, and the height of the pyramid is $OC = 6$. So, $V = \frac{1}{3} \cdot 18 \cdot 6 = 36$. Similarly, plane $x + y + z = 12$ intersects the cube at $(6,6,0)$, $(6,0,6)$, and $(0,6,6)$. So, these vertices along with the point $(6,6,6)$ determine another triangular pyramid that is farthest from the origin and congruent to the first. The volume of the cube is $6^3 = 216$, so the central section is the largest. Its volume is $216 - 36 - 36 = 144$.



15\. **720** Using implicit differentiation to find the slope of the tangent line, $5x^2 + 12y^2 = 83 \longrightarrow 10x + 24yy' = 0 \longrightarrow y' = \frac{-5x}{12y}$ and $y'(a,b) = \frac{-5a}{12b}$. The slope of the line joining (a,b) and $(4.15, 0)$ is $\frac{b}{a-4.15} = \frac{-5a}{12b}$. So, $12b^2 = -5a^2 + 20.75a \longrightarrow 5a^2 + 12b^2 = 20.75a$. Since (a,b) lies on the ellipse, $5a^2 + 12b^2 = 83$. So, $20.75a = 83 \longrightarrow a = 4$. By substitution, $80 + 12b^2 = 83 \longrightarrow 12b^2 = 3 \longrightarrow b^2 = \frac{1}{4} \longrightarrow b = \frac{1}{2}$, only (since (a,b) is in the first quadrant). Thus, $360 \cdot 4 \cdot \frac{1}{2} = 720$.

Mathletics - NMT 2017 **Solutions**

1. **314** If $g(1) = 314$, then $g(2) = 1 - g(1) = 1 - 314 = -313$; $g(3) = 1 - g(2) = 1 + 313 = 314$; $g(4) = 1 - g(3) = 1 - 314 = -313$; $g(5) = 1 - g(4) = 1 + 313 = 314$; ...
So, the values of $g(x)$ alternate between 314 and -313. When x is even, $g(x) = -313$, and when x is odd, $g(x) = 314$. Thus, $g(2017) = 314$.

2. **13** Factoring by the difference of two squares yields $[(10x - 3) + (4x - 9)][(10x - 3) - (4x - 9)] = 0 \longrightarrow (14x - 12)(6x + 6) = 0 \longrightarrow x = \frac{12}{14} = \frac{6}{7}$ or $x = -1$. Thus, the required root is $\frac{a}{b} = \frac{6}{7}$ and $6 + 7 = 13$.

3. **600** Area ΔAEF = Area Square $ABCD$ − Area ΔADE − Area ΔECF − Area ΔABF = $40^2 - (0.5)(40)(20) - (0.5)(20)(20) - (0.5)(40)(20) = 1600 - 400 - 200 - 400 = 600$.

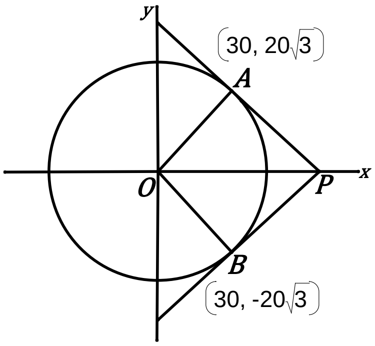
4. **162** The volume of the pool, $V = \pi r^2 h = \pi(9^2)(4) = 1017.876\ \text{ft}^3 \cdot \frac{7.614\ \text{gallons}}{1\ \text{ft}^3} = 7750.108$ gallons. Water is draining at $\frac{2\ \text{gallons}}{\text{hour}} \cdot \frac{24\ \text{hours}}{\text{day}} = \frac{48\ \text{gallons}}{\text{day}}$. So, 7750.108 gallons $\cdot \frac{1\ \text{day}}{48\ \text{gallons}} =$ 161.460 days. Thus, the least number of complete days it takes to drain the pool is 162.

5. **630** The number $20^{17} = (2^2 \cdot 5)^{17} = 2^{34} \cdot 5^{17}$. All divisors of this number are of the form $2^a \cdot 5^b$ where a and b are integers, and $0 \leq a \leq 34$ and $0 \leq b \leq 17$. Since there are 35 possible values of a and 18 possible values of b, the number of possible divisors of $2^{34} \cdot 5^{17}$ is $35 \cdot 18 = 630$.

6. **42** Each term of $\{S_n\}$ is a finite arithmetic series with first term 7 and common difference 2. So, $S_n = \frac{n}{2}(7 + 7 + 2(n - 1)) = \frac{n}{2}(2n + 12) = n^2 + 6n < 2017 \longrightarrow n^2 + 6n - 2017 < 0$. Using the quadratic formula, or a graph, or a calculator table, the largest integer root is $n = 42$.
Alternatively, since the sum of the first n consecutive odd integers is n^2, then $S_n = (n + 3)^2 - 9$. Solving $(n + 3)^2 - 9 < 2017 \longrightarrow (n + 3)^2 < 2026 \longrightarrow |n + 3| < \sqrt{2026} \longrightarrow -48.011 < n < 42.011$. Thus, the required value of n is 42.

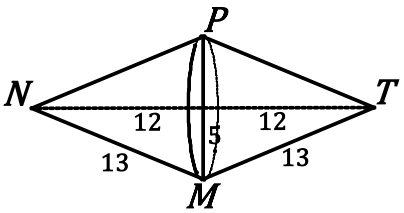
7. **52** By inspection, we see that the point (1, 1) lies on the line $24x + 7y = 31$. The distance from this point to the line $24x + 7y = 1331$ can be found using the formula $d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$, where (x_0, y_0) is the given point and $ax + by + c = 0$ is the given line. So, $d = \frac{|24(1) + 7(1) - 1331|}{\sqrt{24^2 + 7^2}} = \frac{1300}{\sqrt{625}} = \frac{1300}{25} = 52$.
Alternatively, the two given lines are parallel, so the distance between them is measured by the length of a segment perpendicular to both lines. An equation of the line through (1, 1) that is perpendicular to both lines is $y - 1 = \frac{7}{24}(x - 1)$. This line intersects $24x + 7y = 1331$ at $\left(\frac{1273}{25}, \frac{389}{25}\right)$. So, the distance between this point and (1, 1) is $\sqrt{\left(\frac{1273}{25} - 1\right)^2 + \left(\frac{389}{25} - 1\right)^2} = 52$.

8. **70** Since $x = 30$, $30^2 + y^2 = 2100 \longrightarrow y = \pm\sqrt{1200} = \pm 20\sqrt{3}$. The coordinates of A and B are $(30,\ 20\sqrt{3})$ and $(30, -20\sqrt{3})$, respectively. In the given circle O, the slope of radius $\overline{OA}$ is $\frac{20\sqrt{3}}{30} = \frac{2\sqrt{3}}{3}$, so the slope of tangent $\overleftrightarrow{PA}$ is $-\frac{3}{2\sqrt{3}}$, and an equation of $\overleftrightarrow{PA}$ is $y - 20\sqrt{3} = -\frac{3}{2\sqrt{3}}(x - 30)$. Since point P is equidistant from points A and B, it must lie on the x-axis. So, when $y = 0$, $-20\sqrt{3} = \dfrac{-3x}{2\sqrt{3}} + \dfrac{90}{2\sqrt{3}} \longrightarrow -120 = -3x + 90 \longrightarrow 3x = 210 \longrightarrow x = 70$.



9. **34** The probability of rolling 112233445566 in that order is $\left(\frac{1}{6}\right)^{12}$. The number of permutations of these 12 objects with 6 repeated pairs is $\frac{12!}{(2!)^6}$. Thus, the required probability is $10{,}000 \cdot \left(\frac{1}{6}\right)^{12} \cdot \frac{12!}{(2!)^6} = 34.382 \ldots \approx 34$.

10. **200** The solid created consists of two cones joined at their bases. The altitude from point M of ΔNMT bisects $\overline{NT}$ into two segments of length 12. By the Pythagorean Theorem, the length of the altitude is 5. This altitude is also a radius of each of the cones, and since $\overline{NT}$ is the axis perpendicular to the cones' bases, each cone has a height of 12. So, $V = 2\left(\frac{1}{3}\pi r^2 h\right) = 2\left(\frac{1}{3}\pi(5^2)(12)\right) = 200\pi$. Thus, $k = 200$.



Team Problem Solving - NMT 2017 Solutions

1. **49** Since $a \ @\ b \ @\ c = \frac{a+b+c}{3}$, and we want the greatest integer value for $34 \ @\ 16 \ @\ c$, we need c to be the greatest possible two-digit number that results in a sum, $50 + c$, that is a multiple of 3. When $c = 97$, the sum is 147 and the mean is 49.

2. **30** Reduce the fraction $\frac{279}{261}$ to $\frac{31}{29}$. It might become obvious that $p = 30$ and $q = 1$. If not, then multiply both sides of the equation by $29(p - q)$ to eliminate the fractions: $29p + 29q = 31p - 31q \longrightarrow 60q = 2p \longrightarrow \frac{p}{q} = \frac{60}{2} = 30$. Alternatively, first divide each term of the left side of the original equation by q and then solve for $\frac{p}{q}$.

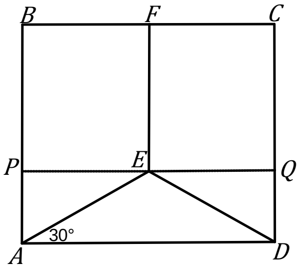
3. **12** The coordinates of the midpoints of the sides of a triangle are the averages of the coordinates of the vertices of the triangle. So, $\frac{d+f}{2} = -2$, $\frac{f+h}{2} = 3$, and $\frac{h+d}{2} = 1$. Thus, $d + f + h = -2 + 3 + 1 = 2$. Similarly, $e + g + k = 5 + 4 + 1 = 10$. So, the sum of all 6 numbers is $2 + 10 = 12$.

4. **30** When a parabola is tangent to the x-axis, the discriminant of its defining function must be 0. Therefore, $b^2 - 4(5)(45) = 0 \longrightarrow b^2 = 900 \longrightarrow b = \pm 30$. Since the graph is tangent to the positive x-axis, $b = 30$.

5. **237** Let $\log_{\frac{1}{b}}\left(\frac{1}{x}\right) = w$ and then convert the two equations into equations involving exponents: $\log_b(x) = 237 \longrightarrow b^{237} = x$ and $\log_{\frac{1}{b}}\left(\frac{1}{x}\right) = w \longrightarrow \left(\frac{1}{b}\right)^w = \frac{1}{x} \longrightarrow b^w = x$. Thus, $w = 237$.

6. **190** Rewrite the equation in terms of powers of 2 and 5: $(2^2 \cdot 5)^{x+y} \cdot (5^2)^{x-2y} = (2^2)^4 \cdot (2 \cdot 5)^{30}$. Apply the distributive property for exponents over multiplication: $2^{2x+2y} \cdot 5^{x+y} \cdot 5^{2x-4y} = 2^8 \cdot 2^{30} \cdot 5^{30} \longrightarrow 2^{2x+2y} \cdot 5^{3x-3y} = 2^{38} \cdot 5^{30}$. Thus, $2x + 2y = 38 \longrightarrow x + y = 19$ and $3x - 3y = 30 \longrightarrow x - y = 10$. So, $x^2 - y^2 = (x + y)(x - y) = 19 \cdot 10 = 190$.

7. **600** Draw $\overline{PQ} \parallel \overline{AD}$ through E. It is easy to see that the area of rectangle $APQD$ is twice the area of ΔAED. Since $m\measuredangle EAD = m\measuredangle PEA = 30°$, $AE = 2AP = EF$. The area of rectangle $PBCQ$ is twice the area of rectangle $APQD$, so the area of $ABCD = 6$ times the area of $\Delta AED = 6(100) = 600$ square units.



8. **112** The slope of the line containing the two given points is $\frac{618-63}{909-21} = \frac{555}{888} = \frac{5}{8}$. This means that for every 5 units added to the y-coordinate, we need to add 8 units to the respective x-coordinate to stay on the line. There are $\frac{555}{5} = 111$ steps of 5 that will take you from 63 to 618. Since the question asks for the number of points on the segment that have integer coordinates, we need to add the starting point to 111 to get a total of 112 points.

9. **20** Since the diagonals of a rhombus bisect each other, the point $(5, 5)$ is the midpoint of both diagonals. So, $\frac{p+r}{2} = 5$ and $\frac{q+s}{2} = 5 \longrightarrow p + r = 10$ and $q + s = 10 \longrightarrow p + q + r + s = 20$.

10. **198** The mean of the set of consecutive integers is $\frac{1881}{19} = 99$. The sum of the largest integer in the set and the smallest integer in the set is therefore twice the mean, or 198.

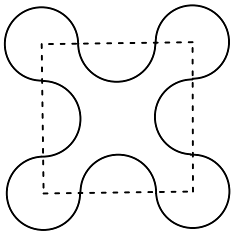
11. **35** The factors of 495 are 5, 9, and 11, so d must be 5. A number that is divisible by 11 has the property that alternating the signs of the individual digits and then adding the signed numbers together must result in a sum that is a multiple of 11. Therefore, $1-a+b-c+5$ must be a multiple of 11. The digits available for a, b, and c are 2, 3, and 7, so $b=3$. Since we want the greatest possible other factor, let $a=7$ and $c=2$. Finally, $\frac{17325}{495}=35$.

12. **2** Since $x^2-y^4=3\cdot 7\cdot 29 \longrightarrow (x-y^2)(x+y^2)=3\cdot 7\cdot 29$, and since $x-y^2<x+y^2$, the value of $x-y^2$ is either 1, 3, 7, or $3\cdot 7=21$. Examine each case: (a) If $x-y^2=1$, then $x+y^2=609 \longrightarrow 2x=610 \longrightarrow x=305 \longrightarrow y^2=304$ and y is not an integer; (b) If $x-y^2=3$, then $x+y^2=203 \longrightarrow 2x=206 \longrightarrow x=103 \longrightarrow y^2=100$ and $y=10$; (c) If $x-y^2=7$, then $x+y^2=87 \longrightarrow 2x=94 \longrightarrow x=47 \longrightarrow y^2=40$ and y is not an integer; (d) If $x-y^2=21$, then $x+y^2=29 \longrightarrow 2x=50 \longrightarrow x=25 \longrightarrow y^2=4$ and $y=2$. Thus, there are only 2 solutions: $(103, 10)$ and $(25, 2)$.

13. **36** Since $(a+b)^3=a^3+b^3+1404 \longrightarrow a^3+3a^2b+3ab^2+b^3=a^3+b^3+1404 \longrightarrow 3a^2b+3ab^2=1404 \longrightarrow 3ab(a+b)=1404$. It is given that $a+b=13$, so $3ab(13)=1404 \longrightarrow ab=36$.

14. **144** Let $x=m(y+z)$ and $y=n(z-x)$ for some integers m and n. Substitute the given values to create the equations $20=m(-15+25)$ and $-15=n(25-20) \longrightarrow m=2$ and $n=-3$. Use these values and the information that $z=5$ to get $x=2(y+5)$ and $y=-3(5-x)$. Solve this system of equations to get $x=4$ and $y=-3$. Thus, $(xy)^2=(-12)^2=144$.

15. **17** Draw a square so that the semi-circles and quarter-circles become visible. The area of the shape we are looking for is the area of the square minus the area of 4 quarter-sectors plus the area of 4 semi-circles. Since the radius of each circular arc is 1, the square has side 4 and its area is $4^2=16$. The area of each quarter-sector is $\frac{\pi(1)^2}{4}$ and the area of each semi-circle is $\frac{\pi(1)^2}{2}$. The total area is $16-4\left(\frac{\pi}{4}\right)+4\left(\frac{\pi}{2}\right)=16+\pi$. Thus, $a=16$, and $b=1$, so $a+b=17$.



16. **180** Given that $\frac{1}{5}=\frac{1}{a}+\frac{1}{b} \longrightarrow \frac{1}{b}=\frac{1}{5}-\frac{1}{a}=\frac{a-5}{5a} \longrightarrow b=\frac{5a}{a-5}$. Examine a table of values for a and b with integers a and b both greater than 5:

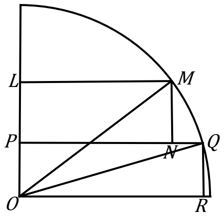
a	6	7	8	9	10	11	...	30
b	30	X	X	X	10	X	...	6

The "X" represents non-integer values. Notice that as a increases, b decreases. The only integers satisfying the equation are 6 and 30, or 10 and 10. Since $a \neq b$, the values are 6 and 30, so $ab=180$.

17. **116** A theorem to count divisors of a positive integer is as follows: If $N=p^{\alpha}q^{\beta}\ldots r^{\gamma}$, where p, q, and r are the prime factors of N, and α, β, and γ are positive integers, then N has $(\alpha+1)(\beta+1)\ldots(\gamma+1)$ divisors. In order for N to have exactly six divisors, then N is either of the form p^5 or pq^2. Trial and error yields: $2^5=32<100$; $3^5=243$; $7\cdot 5^2=175$; $3\cdot 7^2=147$; $13\cdot 3^2=117$; $29\cdot 2^2=116$. Further attempts to factor the integers between 100 and 116 yield an incorrect number of divisors.

18. **26** Convert each number on the right side of the equation to a power of 2: $4^{12}=(2^2)^{12}=2^{24}$, $8^8=(2^3)^8=2^{24}$, $16^6=(2^4)^6=2^{24}$, and $64^4=(2^6)^4=2^{24}$. It follows that $2^{24}+2^{24}+2^{24}+2^{24}=4(2^{24})=(2^2)(2^{24})=2^{26}$. So, $n=26$.

19.**160** Apply the Pythagorean Theorem in ΔORQ to find that the radius of the quarter-circle $OQ = OM = 25$. In ΔOML, if we let $LP = 2x$ and $LM = 5x$, then by the Pythagorean Theorem again, $(5x)^2 + (2x + 7)^2 = 25^2 \longrightarrow 25x^2 + 4x^2 + 28x + 49 = 625 \longrightarrow 29x^2 + 28x - 576 = 0$. Factor or apply the quadratic formula to find that the only positive value for x is 4. Therefore, the sides of the rectangle are $2(4) = 8$ and $5(4) = 20$. So, the area of the rectangle is $8 \cdot 20 = 160$ sq. units.



20. **120** Since the expressions must be consecutive odd integers, their difference must be 2. Therefore, either $(2x^2 - 3x + 6) - (x^2 + 3x - 1) = 2$ or $(2x^2 - 3x + 6) - (x^2 + 3x - 1) = -2$. Simplifying the first equation yields $x^2 - 6x + 5 = 0 \longrightarrow (x - 5)(x - 1) = 0 \longrightarrow x = 5$ or 1. From the second equation, $x^2 - 6x + 9 = 0 \longrightarrow (x - 3)^2 = 0 \longrightarrow x = 3$. When $x = 5, 1,$ and 3, the values of the expressions, in pairs, are: 39 and 41, 3 and 5, and 15 and 17, respectively. So, the sum of these three pairs is: $39 + 41 + 3 + 5 + 15 + 17 = 120$..