Grade Level 9 - NMT 2016 **Solutions**

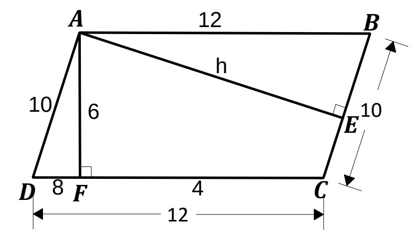
1. **55** $(3,2,4)^* = \frac{3(2)+2(4)+4(3)}{9+4+16} = \frac{26}{29}$. So, the required sum is $26 + 29 = 55$.

2. **24** If $A = 12C$, then $\pi r^2 = 12(2\pi r)$. Since $r > 0$, divide both sides by πr to get $r = 24$.

3. **168** If the dimensions of the rectangular solid are l, w, and h, then $lw = 28 = 4 \cdot 7$, $wh = 24 = 4 \cdot 6$, and $lh = 42 = 6 \cdot 7$. The product of the three equations is: $l^2w^2h^2 = 4^2 \cdot 6^2 \cdot 7^2$. Thus, the volume of the solid, $V = lwh = 4 \cdot 6 \cdot 7 = 168$.

4. **56** $0.25(x-8) \geq 0.5x - 16 \longrightarrow 0.25x - 2 \geq 0.5x - 16 \longrightarrow 14 \geq 0.25x \longrightarrow 56 \geq x \longrightarrow x \leq 56$.

5. **720** Opposite sides of a parallelogram are congruent, so $DC = 12$ and $DF = 8$. By the Pythagorean Theorem, $AD = \sqrt{6^2 + 8^2} = 10$, so $BC = 10$. The area of parallelogram $ABCD = DC \cdot AF = BC \cdot AE$. Thus, $12 \cdot 6 = 10 \cdot h \longrightarrow$ $10h = 72 \longrightarrow 100h = 720$.



6. **16** If the first, second, and third integers are represented by x, y, and z, respectively, then $\left(\frac{3}{4}\right)x = \left(\frac{6}{5}\right)y = \left(\frac{9}{2}\right)z$ and $y - z = 44$. Since $z = y - 44$, $\left(\frac{6}{5}\right)y = \left(\frac{9}{2}\right)(y - 44) \longrightarrow y = 60$. Now, $\left(\frac{3}{4}\right)x = \left(\frac{6}{5}\right)(60) \longrightarrow x = 96$. Finally, $60 - z = 44 \longrightarrow$ $z = 16$. Thus, the smallest of the three integers is 16.

7. **11** Let $x - 1$, x, and $x + 1$ be the three consecutive integers. So,
$(x+1)^3 - x^3 = x^3 - (x-1)^3 + 66 \longrightarrow$
$x^3 + 3x^2 + 3x + 1 - x^3 = x^3 - x^3 + 3x^2 - 3x + 1 + 66 \longrightarrow 6x = 66 \longrightarrow x = 11$.
Thus, the integers are 10, 11, and 12, and the median integer is 11.

8. **75** Using degree measure, let x, $90 - x$, and $180 - x$ be the angle, its complement, and its supplement, respectively. So, $(90 - x)^2 = 2(180 - x) + 15 \longrightarrow$
$8100 - 180x + x^2 = 360 - 2x + 15 \longrightarrow x^2 - 178x + 7725 = 0 \longrightarrow$
$(x - 75)(x - 103) = 0 \longrightarrow x = 75$ or $x = 103$. Since $103 > 90$, $x = 75$.

9. **3** The units digits of powers of 3, starting with $3^1, 3^2, 3^3, 3^4, \ldots$, repeat in a cycle of 4. The pattern is: 3, 9, 7, 1, 3, 9, 7, 1,… . Similarly, the units digits of the powers of 17, starting with 17^1 have a period of 4 with the pattern 7, 9, 3, 1, … . Additionally, the units digits of the powers of 29 repeat in a cycle of 2, i.e. 9, 1, 9, 1, … . Since the remainders of $\frac{2015}{4}$, $\frac{2016}{4}$, and $\frac{2017}{2}$ are 3, 0 and 1, respectively, the units digits of 3^{2015}, 17^{2016}, and 29^{2017} are 7, 1, and 9, in that order. The product of these digits is 63, so the units digit of the given product is 3.

10. **1** If x is 10% larger than y, then $x = 1.1y$. If y is 20% larger than z, then $y = 1.2z$. If z is 25% smaller than w, then $z = 0.75w$. If x is p% smaller than w, then $x = (1 - 0.01p)w$. So, $x = (1.1)(1.2)(0.75)w \longrightarrow x = 0.99w = (1 - 0.01p)w \longrightarrow 0.99w = w - 0.01pw \longrightarrow$ $0.01pw = 0.01w \longrightarrow p = 1$.

11. **60** Since $899 = 900 - 1$, we can factor it as the difference of two squares: $(30 - 1)(30 + 1) = (29)(31)$. Since 29 and 31 are both prime, their sum is 60.

12. **398** If the first term is 2 and the common difference is d, then the first 5 terms of the arithmetic sequence can be represented by: $2,\ 2+d,\ 2+2d,\ 2+3d,\ 2+4d$, where $d \neq 0$. If the first, second, and fifth terms form a geometric sequence, then, $\frac{2+d}{2} = \frac{2+4d}{2+d} \longrightarrow$ $d^2 + 4d + 4 = 4 + 8d \longrightarrow d^2 - 4d = 0 \longrightarrow d = 4$ or $d = 0$ (reject). So, the arithmetic sequence is 2, 6, 10, 14, 18, … . Using $a_n = a_1 + d(n-1)$, the hundredth term, $a_{100} = 2 + 4(100-1) = 398$.

13. **7** Let $D_1 =$ distance from the fan's current location to his home, $D_2 =$ distance from the fan's current location to the stadium, and $D_3 =$ distance from the fan's home to the stadium. Let R be his walking rate, and $7R$ be his riding rate. Using Distance $=$ Rate $\cdot$ Time, $T_1 = \frac{D_1}{R}$, $T_2 = \frac{D_2}{R}$, and $T_3 = \frac{D_3}{7R}$. If $T_1 + T_3 = T_2$, then $\frac{D_1}{R} + \frac{D_3}{7R} = \frac{D_2}{R}$. Since $D_1 + D_2 = D_3$, substitute to get $\frac{D_1}{R} + \frac{D_1 + D_2}{7R} = \frac{D_2}{R}$. So, $7D_1 + D_1 + D_2 = 7D_2 \longrightarrow 8D_1 = 6D_2 \longrightarrow \frac{D_1}{D_2} = \frac{3}{4}$. Therefore, the required sum is 7.

14. **333** Since $17 \cdot 13 = 221$, the checkerboard is numbered from 1 to 221. Using both numbering schemes, the numbers in the first, last, and middle squares are the same. The middle of the board is the median of $1, 2, 3, \ldots, 221$, which is 111. Thus, the required sum is $1 + 111 + 221 = 333$.

15. **11** We need to consider only whether the number on each top face is even or odd (E or O). We need not be concerned with the actual number. Thus, there are only two possibilities (even or odd) for each of the three top faces. So, there are $2^3 = 8$ possibilities in all. Of these possibilities, outcome (O, O, O) is odd for both sum and product. So, we exclude this from our sample space which consists of all outcomes for which the product is even. There are 7 of these: (O, O, E), (O, E, O), (E, O, O), (O, E, E), (E, O, E), (E, E, O), and (E, E, E). For an outcome to have an even sum as well as an even product, either zero odds or two odds are needed. So, the 4 successful outcomes are (O, O, E), (O, E, O), (E, O, O), and (E, E, E). Therefore, the required probability is $\frac{4}{7}$, and the required sum is 11.

Grade Level 10 - NMT 2016 **Solutions**

1. **104** In a rhombus, the diagonals are perpendicular bisectors of each other. So, each of the four interior non-overlapping triangles of the rhombus is a right triangle with legs of 10 and 24. By the Pythagorean Theorem, a side of the rhombus is $\sqrt{10^2 + 24^2} = 26$, so the perimeter is 104.

2. **21** Performing the transformations from right to left, $D_2(5,2) = (10,4)$, and then $T_{-3,-1}(10,4) = (7,3) = (a,b)$. Thus, $ab = 21$.

3. **30** The prime power decomposition of 16,380 is $2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$. Thus, the distinct prime factors of 16,380 are 2, 3, 5, 7 and 13. The required sum is $2 + 3 + 5 + 7 + 13 = 30$.

4. **12** In an arithmetic sequence, the differences between consecutive terms are equal. Therefore, $(5x+2) - (3x-7) = (9x-13) - (5x+2) \rightarrow 2x + 9 = 4x - 15 \rightarrow x = 12$.

5. **2** Let x be the required number. Thus, $\frac{3+x}{5-x} = \frac{5}{3} \rightarrow 9 + 3x = 25 - 5x \rightarrow x = 2$.

6. **18** If two secants are drawn to a circle from a common external point, then the products of the length of each secant and the length of its external segment are equal.
Thus, $AC \cdot AB = AE \cdot AD$. So, $9 \cdot 6 = AE \cdot 3 \rightarrow AE = 18$.

7. **8** Two vertices of a convex polygon determine either a diagonal or a side of the polygon. The number of ways of choosing two vertices from n vertices is $\binom{n}{2}$. However, this number also includes the n sides of the polygon. So, the number of diagonals is $\binom{n}{2} - n = 20 \rightarrow \frac{n(n-1)}{2} - n = 20 \rightarrow n^2 - 3n - 40 = 0 \rightarrow n = 8$ or -5. Thus, the polygon has 8 sides. Alternatively, we can use the fact that a convex polygon with n sides has $\frac{n(n-3)}{2}$ diagonals.

8. **198** The value of x for which $2x - 5 = 7$ is $x = 6$. Therefore, $f(7) = 8 \cdot 6^2 - 26 \cdot 6 + 66 = 198$.

9. **60** The centroid of a triangle is the intersection of the medians. The centroid divides each median into two segments whose lengths are in a 2:1 ratio. So, the length of the median to side $\overline{BC}$ is 7.5, and since the triangle is isosceles, this median is also an altitude. So, using $A = \frac{1}{2}bh$, the area of $\Delta ABC = \frac{1}{2} \cdot 16 \cdot 7.5 = 60$.

10. **171** Use the distance formula to calculate the length of the diameter of the circle and the diagonal of the square: $d = \sqrt{144 + 25} = 13$. Then, since the length of a diagonal of a square equals the length of a side times $\sqrt{2}$, the length of each side is $\frac{13}{\sqrt{2}}$. The area of the square is $\frac{169}{2}$ and the required sum is 171.

11. **20** Note that in right ΔDBC, the length of leg $\overline{BC}$ is $\sqrt{3}$ times the length of leg $\overline{DC}$. Thus, ΔDBC is a 30-60-90 triangle and $m\measuredangle DBC = 30$. Since $\measuredangle AEB$ and $\measuredangle BEC$ are supplementary, $m\measuredangle BEC = 130$. Using ΔBEC, $m\measuredangle ACB = 180 - (130 + 30) = 20$.

12. **17** Rewrite the given equation by multiplying by the LCD, $x^2 - 9$ to get:
$3x^2 - 27 - (5x + 3)(x + 3) = x^2 - 3x \rightarrow x^2 + 5x + 12 = 0$. The product of the roots of this quadratic equation is 12 and the sum of the roots is -5. Thus, the required difference is 17.

13. **144** The lengths of the sides of ΔABC are 10, 12, and 14. To find the area of ΔABC we can use Heron's formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$ where s represents the semi-perimeter, and a, b, and c represent the lengths of the sides. Thus, $A = \sqrt{18 \cdot 8 \cdot 6 \cdot 4} = 24\sqrt{6}$. The required product is 144.

14. **240** If we treat the two consecutive O's as a single unit, and we consider the repeating S's, there are $\frac{6!}{2!} = 360$ arrangements. Included in our count are 5! = 120 arrangements that have the two consecutive S's. We use 5! because the two consecutive O's and the two consecutive S's can each be treated as a single unit. Thus, the required number of arrangements is
360 – 120 = 240.

15. **31** Since each integer pair is not relatively prime, each pair shares a prime factor. Since all three integers are relatively prime, there is no prime factor that is a prime factor of each of the integers. Use these facts to test the smallest primes. Thus, the prime factorizations are $2 \cdot 3$, $3 \cdot 5$, and $2 \cdot 5$. So, the required sum is $6 + 15 + 10 = 31$.

Grade Level 11 - NMT 2016 **Solutions**

1. **13** The sum of the digits of 1287 is divisible by 9; so, 1287 is also divisible by 9. Thus, $1287 = 9 \cdot 143 = 9(144 - 1) = 9(12^2 - 1^2) = 9(12 - 1)(12 + 1) = 3^2 \cdot 11 \cdot 13$. Therefore, the largest prime factor of 1287 is 13.

2. **288** Multiply numerator and denominator by the conjugates of the factors in the denominator: $\frac{2}{(\sqrt{19}-\sqrt{3})(\sqrt{41}-\sqrt{5})} \cdot \frac{(\sqrt{19}+\sqrt{3})(\sqrt{41}+\sqrt{5})}{(\sqrt{19}+\sqrt{3})(\sqrt{41}+\sqrt{5})} = \frac{2(\sqrt{19}+\sqrt{3})(\sqrt{41}+\sqrt{5})}{(19-3)(41-5)} = \frac{2(\sqrt{19}+\sqrt{3})(\sqrt{41}+\sqrt{5})}{(16)(36)} = \frac{(\sqrt{19}+\sqrt{3})(\sqrt{41}+\sqrt{5})}{288}$. Thus, the new denominator is 288.

3. **3** Since $f(g(x)) = x$, f and g are inverse functions. This means that an input of function g is the output of function f. So, to find $g(14)$, we solve $f(x) = \frac{5x-1}{x-2} = 14$ to get $x = 3$.

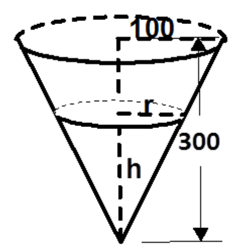
4. **61** By factoring the difference of two cubes, $125x^3 - 64 = (5x - 4)(25x^2 + 20x + 16)$. Thus, $a = 25$, $b = 20$, $c = 16$, and the required sum, $a + b + c = 61$.

5. **29** Let the number of students in the class be n. Then, we know that $n + 1$ is divisible by the least common multiple of 6 and 10. So, $30 \mid (n + 1)$. Therefore, $n = 29$. Alternatively, If a is the number of groups of 6 students, and b is the number of groups of 10 students, then $n = 6a + 5$ and $n = 10b + 9$. So, $6a + 5 = 10b + 9$. The least positive integral values of a and b for which this equation is satisfied are $a = 4$ and $b = 2$. Thus, $n = 29$.

6. **0** If the given sequence is arithmetic, then $k - 1 = 2k + a - k \longrightarrow a = -1$. If the given sequence is geometric, then $\frac{k}{1} = \frac{2k+a}{k} \longrightarrow k^2 = 2k + a \longrightarrow k^2 - 2k + 1 = 0$. So, $k = 1$ and the required sum is $k + a = 1 + (-1) = 0$. Alternatively, note that if a given sequence is both arithmetic and geometric, then its terms are identical. So, for the sequence $1, k, 2k + a$ to have identical terms, $k = 1$ and $2k + a = 1 \longrightarrow a = -1$. So, $k + a = 0$.

7. **0** On the complex plane, multiplying a complex number by i is equivalent to a $90°$ counterclockwise rotation about the origin of the vector representing the complex number. Then, since the sum of any four consecutive terms is 0, and 2016 is a multiple of 4, the required sum is 0. Alternatively, we can compute the first 4 terms and find their sum: $a_1 = 2 - 3i$; $a_2 = 2i - 3i^2 = 2i + 3$; $a_3 = 2i^2 + 3i = -2 + 3i$; $a_4 = -2i + 3i^2 = -2i - 3$. Thus, $a_1 + a_2 + a_3 + a_4 = 0$.

8. **154** If the ratio of the volumes of two similar solids is $a : b$, then the ratio of the corresponding segment lengths of these solids is $\sqrt[3]{a} : \sqrt[3]{b}$. Since the ratio of the volumes is $2 : 1$, the ratio of the depths is $\sqrt[3]{2} : \sqrt[3]{1}$. If h represents the depth of the water that remains in the cup, then $\frac{\sqrt[3]{2}}{1} = \frac{300}{h} \longrightarrow$ $h = \frac{300}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}} = \frac{300\sqrt[3]{4}}{2} = 150\sqrt[3]{4}$. Thus, $p = 150$, $q = 4$, and $p + q = 154$.

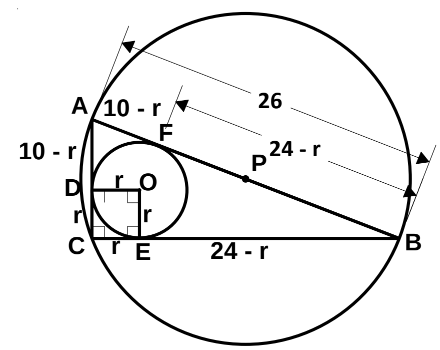


Alternatively, let V_F and V_{HF} represent the volumes of the full cup and the half-full cup, respectively. Using the formula for the volume of a cone, $V_F = \left(\frac{1}{3}\right)\pi(100)^2(300) = \pi(100)^3$. By similar triangles, $\frac{r}{100} = \frac{h}{300} \longrightarrow r = \frac{h}{3}$. So, $V_{HF} = \frac{\pi(100)^3}{2} = \left(\frac{1}{3}\right)\pi\left(\frac{h^3}{9}\right) = \frac{\pi h^3}{27} \longrightarrow \frac{27(100)^3}{2} = h^3 \longrightarrow h = \frac{3(100)}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}} = \frac{300\sqrt[3]{4}}{2} = 150\sqrt[3]{4}$ Thus, $150 + 4 = 154$.

9. **13** Since a and b are integer solutions, the given inequality is satisfied when $x^2 - x - 6 = 0$. So, $(x-3)(x+2) = 0$, and the two integer solutions are 3 and -2. Thus, $a^2 + b^2 = (3)^2 + (-2)^2 = 13$.

10. **65** To find the x-intercept, solve $\log_2(\log_3(\log_4(x-1))) = 0$. Using the definition of a logarithm, $\log_3(\log_4(x-1)) = 2^0 = 1$. Similarly, $\log_4(x-1) = 3^1 = 3$ and $x - 1 = 4^3 = 64$. So, $x = 65$.

11. **185** Since 10-24-26 is a Pythagorean triple, the given triangle is a right triangle. So, the hypotenuse of the triangle is the diameter of the circumscribed circle, P, and the radius of the circumscribed circle is 13. To find the radius, r, of the inscribed circle, O, we use the fact that tangent segments drawn to a circle from an external point are congruent and that $OECD$ is a square. So, $10 - r + 24 - r = 26 \longrightarrow r = 4$. Thus, the ratio of the radii is $13:4$, and the ratio of the areas of the circles is the square of the ratio of the radii, or $169:16$. Therefore, $a + b = 169 + 16 = 185$.



12. **120** In the expansion, the term containing x^2y^2z is found by computing the number of ways that the factors $(2x)(2x)(y)(y)(z)$ can be arranged. This can be done in $\frac{5!}{2!\cdot 2!} = 30$ ways. So, $kx^2y^2z = 30(2x)^2(y)^2(z) = 120x^2y^2z$. Thus, $k = 120$.

13. **15** $\cos^4(x) - \sin^4(x) = \left(\cos^2(x) + \sin^2(x)\right)\left(\cos^2(x) - \sin^2(x)\right) = \cos(2x) = \frac{2}{3}$. Since x is acute, $\sin(2x)$ is positive. So, $\sin(2x) = \sqrt{1 - \cos^2(2x)} = \sqrt{1 - \frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$. Thus, $m = 5$, $n = 3$ and $mn = 15$.

14. **961** From the second given equation, $g(n) = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$. So, from the first given equation, $f(2n) = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$. Then, again, from the second given equation, $g(2n) = \sqrt{1 - \left(\frac{24}{25}\right)^2} = \frac{7}{25}$. Therefore, again, from the first given equation, $f(4n) = 2\left(\frac{24}{25}\right)\left(\frac{7}{25}\right) = \frac{336}{625}$. Thus, $a + b = 336 + 625 = 961$.

15. **41** The given function is symmetric about the lines $x = 1$ and $x = 6$. Since it is given that $(0,0)$ belongs to f, we can generate more roots of f by performing the following line reflections: $(0,0) \xrightarrow{x=1} (2,0) \xrightarrow{x=6} (10,0) \xrightarrow{x=1} (-8,0) \xrightarrow{x=6} (20,0) \xrightarrow{x=1} (-18,0) \xrightarrow{x=6} (30,0)$, etc.
We get more roots if we again start with $(0,0)$ and reverse the order of the line reflections: $(0,0) \xrightarrow{x=6} (12,0) \xrightarrow{x=1} (-10,0) \xrightarrow{x=6} (22,0) \xrightarrow{x=1} (-20,0) \xrightarrow{x=6} (32,0) \xrightarrow{x=1} (-30,0)$, etc.
We now see that the solution set can be separated into two arithmetic sequences: $A = \{-100, \dots, -20, -10, 0, 10, 20, \dots, 100\}$ and $B = \{-98, \dots, -18, -8, 2, 12, \dots, 92\}$.
Using $a_n = a_1 + d(n-1)$, sequence A has 21 terms, and sequence B has 20 terms. Therefore, on the given interval, function f has 41 roots.

Grade Level 12 - NMT 2016 **Solutions**

1. **20** Let P be the original price. $P\left(1-\frac{D}{100}\right)\left(1+\frac{25}{100}\right)=P \longrightarrow \left(1-\frac{D}{100}\right)\left(\frac{5}{4}\right)=1$. Thus, $D=20$.

2. **0** $f(x)=\left(\sin^2(x)+\cos^2(x)\right)^2=1$. So, $f'(x)=0$ for all x.

3. **12** Using trial and error, the ordered pairs $(\pm 5,\ \pm 5)$, $(\pm 1,\ \pm 7)$, and $(\pm 7,\ \pm 1)$ satisfy the given equation. Each one of these 3 pairs yields 4 distinct ordered pairs. So, there are 12 solutions.

4. **9** The numerator can be factored using synthetic division or grouping. By grouping:
$x^5-2x^4+3x^3-6x^2+8x-16=x^4(x-2)+3x^2(x-2)+8(x-2)=$
$(x^4+3x^2+8)(x-2)$. So, $\lim\limits_{x\to 2}\frac{(x^4+3x^2+8)(x-2)}{(x+2)(x-2)}=\lim\limits_{x\to 2}\frac{x^4+3x^2+8}{x+2}=\frac{36}{4}=9$.
Alternatively, L'Hopital's Rule can be used to find the limit.

5. **200** First, change the given equation to: $\log_2\left(\frac{x^3-1}{x-1}\right)=\log_2(40201)$. Therefore, $\frac{x^3-1}{x-1}=40201 \longrightarrow$ $\frac{(x-1)(x^2+x+1)}{x-1}=40201 \longrightarrow x^2+x-40200=0 \longrightarrow (x-200)(x+201)=0 \longrightarrow x=200$, only. We reject $x=-201$ because it is not in the domain of the original equation.

6. **154** Since $6!=720$, all values of $k!$, where $k>6$ are multiples of 360 and have no effect on the final answer. Therefore, it suffices to compute $\sum_{k=0}^{5}k!=1+1+2+6+24+120=154$.

7. **425** Since $\Delta ABC \sim \Delta EDC$, $BC : CD = 240 : 135 = 16 : 9$. Therefore, $16x+9x=200 \longrightarrow x=8$, $BC=128$, and $CD=72$. Use the Pythagorean Theorem twice to compute $AC=272$ and $CE=153$. Thus, $AC+CE=272+153=425$. To simplify the computation, note that the lengths of the sides of both triangles are multiples of the Pythagorean triple 8-15-17. That is, 128-240-272 is **8**(16)-**15**(16)-**17**(16), and 72-135-153 is **8**(9)-**15**(9)-**17**(9).

8. **78** Set S consists of 60 evenly-spaced points around the unit circle. Connecting the consecutive vertices of any evenly-spaced subset of S (of size 3 or more) will produce a regular polygon. Let k represent the number of sides of a regular polygon so formed. Since the vertices of that polygon must also be evenly spaced, k must divide 60. By rotation, such a k-gon can be placed in $\frac{60}{k}$ possible positions. To find the total number of possible polygons, add up all the values of $\frac{60}{k}$, where k is a divisor of 60 greater than or equal to 3. In tabular form:

k	3	4	5	6	10	12	15	20	30	60
$60/k$	20	15	12	10	6	5	4	3	2	1

Thus, the sum of the bottom row produces a total of 78 possible regular polygons.

9. **410** Using $(a+b)^3=a^3+3a^2b+3ab^2+b^3$, $\left(\sqrt{3}+i\right)^3=3\sqrt{3}+3(3)i+3\sqrt{3}(-1)-i=8i$. So, $\left(\sqrt{3}+i\right)^{12}=(8i)^4=4096$. Thus, $\frac{4096}{10}=409.6$ and the closest integer is 410.
Alternatively, $\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=2(\cos 30^\circ+i\sin 30^\circ)$. So, using DeMoivre's Theorem, $\left(\sqrt{3}+i\right)^{12}=2^{12}(\cos 360^\circ+i\sin 360^\circ)=2^{12}=4096$, and 409.6 is closest to 410.

10. **23** Since the product of the roots of the equation is 2016, the eight roots must be (counting multiplicity): 2, 2, 2, 2, 2, 3, 3, and 7. Note that these are the prime factors of 2016. Since A is the sum of the roots of the equation, $A=2+2+2+2+2+3+3+7=23$.

11. **120** There are $\frac{7!}{3!2!2!}$ permutations of ALFALFA. The number of permutations that start with the letter A equals the number of permutations of LFALFA which is $\frac{6!}{2!2!2!}$. Thus, the number of permutations that do *not* start with A is $\frac{7!}{3!2!2!}-\frac{6!}{2!2!2!}=\frac{7\cdot 6!}{3!2!2!}-\frac{3\cdot 6!}{3!2!2!}=\frac{4\cdot 6!}{3!2!2!}=120$.

Alternatively, there are 2 choices for the first letter (L or F), followed by an arrangement of 6 letters whose distribution is of the form 3-2-1, regardless of which letter (L or F) is chosen first. Therefore, the required number of permutations is $2 \cdot \frac{6!}{3!2!} = 120$.

12. **540** This is a geometric series with, $r = \sin\theta$, whose sum is $\frac{a}{1-r} = \frac{1}{1-\sin\theta} = \frac{5}{7} \longrightarrow \sin\theta = -\frac{2}{5}$. Thus, there will be two values of θ, in quadrants III and IV, of the form $180 + k$ and $360 - k$, where $k = Arcsin\frac{2}{5}$. Therefore, the sum of these two values of θ is 540. Note that the actual value of $\sin\theta$ is irrelevant; only the fact that it is negative matters. This fact can be determined by noting that the sum of the series is *less than* the first term, 1.

13. **968** The curves are tangent if they have only one point of intersection. Setting the two equations equal yields $x^2 + b = 88x - x^2 \longrightarrow 2x^2 - 88x + b = 0$. Since there is only one solution, the discriminant must equal zero. Therefore, $88^2 - 4(2)b = 0 \longrightarrow b = 968$.
Alternatively, the two curves must have the same function value and the same derivative value at a particular value of x. Setting the derivatives of the two functions equal gives us $2x = 88 - 2x$, and so the tangency occurs at $x = 22$. Thus, $22^2 + b = 88(22) - 22^2 \longrightarrow b = 4(22)^2 - 2(22)^2 = 2(22)^2 = 968$.

14. **512** Observe that $f(x + x) = f(x) + f(x) + 2\sqrt{f(x)\, f(x)}$. So, for all x, $f(2x) = 4f(x)$. Thus, $f(2) = 4f(1) = 2$, and this leads to $f(4) = 4f(2) = 8$, $f(8) = 4f(4) = 32$, $f(16) = 4f(8) = 128$, and $f(32) = 4f(16) = 512$. Note that $f(x) = \frac{1}{2}x^2$ satisfies all of the given conditions and can be used to find the answer. However, determining this fact can be rather tricky.

15. **143** The derivative, $f'(x) = 2Ax - (A + 1) = 0$ is true only at $x = \frac{A+1}{2A}$. Since f is an upwards-opening parabola, it has its minimum value at this point. The minimum value of f is:
$\min(A) = f\left(\frac{A+1}{2A}\right) = A\left(\frac{A+1}{2A}\right)^2 - (A+1)\left(\frac{A+1}{2A}\right) + 144 = \frac{(A+1)^2}{4A} - \frac{2(A+1)^2}{4A} + 144 = 144 - \frac{(A+1)^2}{4A}$.
Now we need to maximize $\min(A)$, which means that $\frac{(A+1)^2}{4A}$ must be the smallest possible non-negative integer. We try $\frac{(A+1)^2}{4A} = 0$ and reject it because $A < 0$. Next, we try $\frac{(A+1)^2}{4A} = 1$. Thus, $A = 1$ and we meet the requirement that $A > 0$. Therefore, the maximum value of $\min(A)$ is $144 - 1 = 143$.
Alternatively, we can maximize $\min(A)$ by taking its derivative and solving $\min'(A) = 0$ to get $A = 1$ and $\min(1) = 143$.

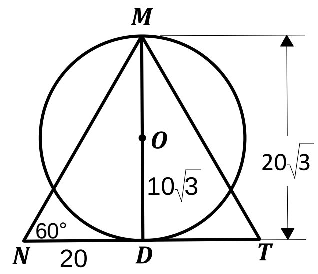
Mathletics - NMT 2016 **Solutions**

1. **20** Add the equations to yield $36x + 36y = 720$. Then divide by 36 to get $x + y = 20$.

2. **13** Point C is $\frac{3}{7}$ of the way from point A to point B. So, $x = -2 + \frac{3}{7}\big(5 - (-2)\big) = 1$ and $y = -3 + \frac{3}{7}\big(11 - (-3)\big) = 3$. Therefore, $10x + y = 13$.

3. **10** The shortest segment from a point to a circle lies on the line joining the given point to the center of the circle. The distance from the center of the circle (5, 1) to $(19, -1)$ is $\sqrt{(19-5)^2 + (-1-1)^2} = \sqrt{200}$. The length of the radius of the circle is $\sqrt{8}$. Since the given point is in the exterior of the circle, the distance from $(19, -1)$ to the circle is $\sqrt{200} - \sqrt{8} = 10\sqrt{2} - 2\sqrt{2} = 8\sqrt{2}$, and the required sum is 10.

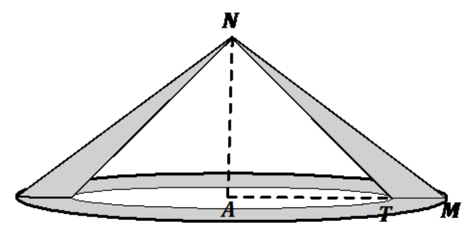
4. **300** The area of an equilateral triangle with side length x is $\frac{x^2\sqrt{3}}{4}$. Solving $\frac{x^2\sqrt{3}}{4} = 400\sqrt{3}$ gives $x = NT = 40$. An altitude of an equilateral triangle divides it into two congruent $30° - 60° - 90°$ triangles, so $ND = 20$ and $MD = 20\sqrt{3}$. Since point O is the midpoint of altitude $\overline{MD}$, the length of the radius of the circle is $10\sqrt{3}$ and the area of the circle is $A = \pi r^2 = \pi\left(10\sqrt{3}\right)^2 = 300\pi$. So, $n = 300$.



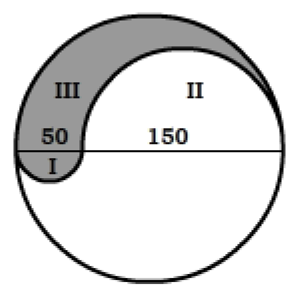
5. **525** Let $x =$ the escalator's speed, $a =$ Andrea's speed when the escalator is not operating, and $d =$ the escalator's length. Three expressions for the length of the escalator are: $d = 70a$, $d = 30(a + x)$, and $d = nx$. So, $30a + 30x = 70a \longrightarrow x = \left(\frac{4}{3}\right)a$. Since $a = \frac{d}{70}$, $x = \left(\frac{4}{3}\right)\left(\frac{d}{70}\right) = \frac{2d}{105}$. Thus, $n = \frac{d}{x} = \frac{d}{\frac{2d}{105}} = 52.5$. So, $10n = 525$.

6. **3** By listing the first few powers of 2: $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, we see that the units digits repeat with a period of 4. The pattern is, $2,4,8,6,2,4,8,6,\ldots$. Since 2016 is a multiple of 4, the units digit of 2^{2016} matches that of $2^4 = 16$. Powers of 0, 1, and 6 always yield a units digit of 0, 1, and 6, respectively. Therefore, the sum is $6 + 0 + 1 + 6 = 13$, and the required units digit is 3.

7. **243** The resulting solid is a hollowed out cone with a circular base. If we rotate ΔNAM about $\overleftrightarrow{NA}$ we get a cone, and if we rotate ΔNAT about $\overleftrightarrow{NA}$ we get a cone with the same height, but a smaller radius. The volume we seek is the difference between the volumes of these two cones. In ΔNAM, using the Pythagorean triple 8-15-17, the height of both cones is $NA = h = 8$. So now, using Volume of a cone $= \left(\frac{1}{3}\right)\pi r^2 h$, $V = \left(\frac{1}{3}\right)\pi(MA)^2 h - \left(\frac{1}{3}\right)\pi(AT)^2 h = \left(\frac{1}{3}\right)\pi[(15)^2(8) - (14)^2(8)] \approx 243$.



8. **4** Area(whole circle)$= 100^2\pi = 10000\pi$.
Area(I) $= \frac{25^2\pi}{2} = 312.5\pi$; Area(II) $= \frac{75^2\pi}{2} = 2812.5\pi$;
Area(III) = Area(whole semicircle) − Area(II)$= \frac{100^2\pi}{2} - \frac{75^2\pi}{2} = 2187.5\pi$.
Shaded area = Area(I) + Area(III) $= 312.5\pi + 2187.5\pi = 2500\pi$.
Unshaded area = Area(whole circle) − Shaded area $= 10000\pi - 2500\pi = 7500\pi$.
So, the required ratio is $\frac{2500}{7500} = \frac{1}{3}$. Thus, $p + q = 4$.



9. **285** P(Brandon wins all 5 games) $= (1)(0.8)(0.6)(0.4)(0.2) = 0.0384$
P(Brandon wins all but game 5) $= (1)(0.8)(0.6)(0.4)(0.8) = 0.1536$
P(Brandon wins all but game 4) $= (1)(0.8)(0.6)(0.6)(0.2) = 0.0576$
P(Brandon wins all but game 3) $= (1)(0.8)(0.4)(0.4)(0.2) = 0.0256$
P(Brandon wins all but game 2) $= (1)(0.2)(0.6)(0.4)(0.2) = 0.0096$
The sum of these probabilities is 0.2848, so $1000p = 284.8 \approx 285$.

10.**144** Let r_n be the number of valid arrangements of n lights with the first light red, and let b_n be the number of valid arrangements of n lights with the first light blue. The first few valid arrangements are as follows: $r_2 = 1$ (rb) and $b_2 = 2$ (br, bb); $r_3 = 2$ (rbr, rbb) and $b_3 = 3$ (bbr, brb, bbb). The sequences $\{r_n\}$ and $\{b_n\}$ can be generated recursively by noting that we can place a blue light in front of any sequence, and we can place a red light in front of any sequence starting with blue. This gives us the recursive formulas: $r_{n+1} = b_n$ and $b_{n+1} = r_n + b_n$. Using these, we can make a table:

$\boldsymbol{n}$	2	3	4	5	6	7	8	9	10
$\boldsymbol{r_n}$	1	2	3	5	8	13	21	34	55
$\boldsymbol{b_n}$	2	3	5	8	13	21	34	55	89

After the first few iterations of the formulas, we see that the numbers of arrangements are members of the Fibonacci sequence. So, when $n = 10$, there are $55 + 89 = 144$ valid arrangements of red and blue lights.

Team Problem Solving - NMT 2016 Solutions

1. **113** Since the number midway between two given numbers is their arithmetic mean, $\frac{a}{b} = \frac{1}{2}\left(\frac{3}{7} + \frac{4}{5}\right) = \frac{43}{70}$. Since 43 and 70 are relatively prime, $a = 43$ and $b = 70$ and the required sum is 113.

2. **40** Since the side of the square is 10, its perimeter is 40. Two sides of the triangle have lengths of 8 and 17, so the length of the missing side is $40 - (8 + 17) = 15$. The sides of the triangle are 8-15-17 which is a Pythagorean triple, so the area of the right triangle is $\frac{8 \cdot 15}{2} = 60$. Since the area of the square is 100, it is $100 - 60 = 40$ square units greater than the area of the triangle.

3. **52** Since $16 = 2^4$, the question can be restated as $\frac{2^{56}}{2^4} = 2^m$. Thus, $m = 56 - 4 = 52$.

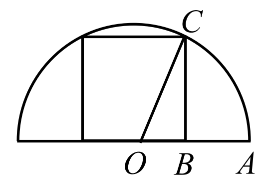
4. **0** The slope of $\overleftrightarrow{AB} = \frac{(10-4)}{(3-1)} = 3$. The slope of $\overleftrightarrow{BC} = \frac{(12-10)}{(5-3)} = 1$. The average of the slopes, $W = 2$. The slope of the line of best fit is $\frac{6}{3} = 2$. So, the difference between W and the slope of the line of best fit is $2 - 2 = 0$.

5. **10** The value of $p = \frac{15+12+5-1}{4} = \frac{31}{4}$ and the value of $q = \frac{-6+0+5+10}{4} = \frac{9}{4}$. The sum $p + q = \frac{31}{4} + \frac{9}{4} = \frac{40}{4} = 10$.

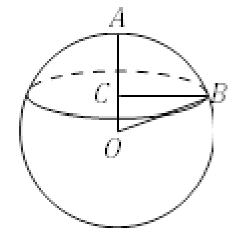
6. **36** Every positive divisor of 2016 must be of the form $2^a \cdot 3^b \cdot 7^c$, where $a, b,$ and c are integers, and $0 \le a \le 5$, $0 \le b \le 2$, and $0 \le c \le 1$. That means that there are 6 possible values for a, 3 for b, and 2 for c. Thus, $6 \cdot 3 \cdot 2 = 36$.

7. **16** The smallest trapezoid with the property that its area and perimeter are numerically equal is a square with sides of length 4. The area and perimeter are both 16. The next larger trapezoid is a rectangle. See if you can find the dimensions of that trapezoid.

8. **77** Since the radius of the semicircle is 10, the area of the semicircle is $\frac{100\pi}{2} = 50\pi$. Let s be the length of a side of the square. So, $BC = s$ and $OB = \frac{s}{2}$. Apply the Pythagorean Theorem to find s: $s^2 + \left(\frac{s}{2}\right)^2 = 10^2$. So $s = 4\sqrt{5}$. The area of the square is $s^2 = 80$, so the required difference is $50\pi - 80 \approx 77$.



9. **169** Draw radii $\overline{OA}$ and $\overline{OB}$ and let each of their lengths be x. Since $AC = 104$, $OC = x - 104$, we apply the Pythagorean Theorem in ΔOCB: $(x-104)^2 + 156^2 = x^2 \longrightarrow x = 169$. [Note: Since $104 = 13 \cdot 8$ and $156 = 13 \cdot 12$ have a common factor of 13, you could work with smaller numbers for the height of the top of the sphere (8), and for the radius of the top of the sphere (12). This leads to a solution of 13 which you would then have to multiply by 13 to get the final answer: $13 \cdot 13 = 169$.]



10. **292** Since the diagonals of a parallelogram bisect each other, the midpoints of the two diagonals must be the same point. This implies that the sum of the x-coordinates of one diagonal must equal the sum of the x-coordinates of the other diagonal. The same is true for the y-coordinates. Therefore, $-12 + 40 = 24 + p \longrightarrow p = 4$, and $45 + 10 = -18 + q \longrightarrow q = 73$. Thus, $pq = 4 \cdot 73 = 292$.

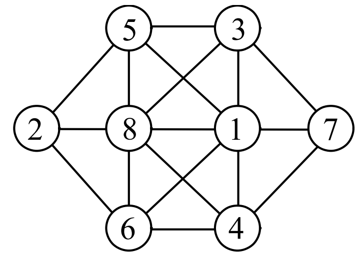
11. **992** The key is to recognize that $6.25\% = \frac{1}{16}$, so you are looking for the largest integer that is less than 1000 and divisible by 16. Note that $\frac{1000}{16} = 62.5$, and $62 \cdot 16 = 992$, so 992 is the largest integer such that when you decrease it by 6.25% you get the integer 930 (which is 62 less than 992).

12. **340** There are 10 possibilities if all three digits are the same. When two digits are the same and the third is different, there are $10 \cdot 9 = 90$ possible arrangements. If all three digits are different, we need to account for two unique clockwise arrangements (i.e. on a circle, the clockwise arrangement 1,2,3 is different than the clockwise arrangement 1,3,2). First determine that there are ${}_{10}C_3 = 120$ combinations of three digits and each combination can appear in two different directions, so there are $120 \cdot 2 = 240$ clockwise arrangements for three different digits. Thus, the total number of arrangements is $10 + 90 + 240 = 340$.

13. **309** Using a TI-84 graphing calculator set to radian mode in a ZDecimal window, we want to solve $\sin(x^r) = \sin(x^\circ)$. Since we're in radian mode, we want the arguments of both sine expressions to be in radians. We know that $x^\circ = \frac{\pi x}{180}$ radians, so we leave the left side of our equation alone and replace the right side with $\sin\left(\frac{\pi x}{180}\right)$. Now we find the zeros/roots/x-intercepts of $Y = \sin x - \sin\left(\frac{\pi x}{180}\right)$. The first positive zero is 3.0877…. We multiply this value by 100 and round to the nearest integer to get 309.
Alternatively, if we use degree mode and the same ZDecimal window, the graph and zeros of $Y = \sin\left(\frac{180x}{\pi}\right) - \sin x$ are identical to the function above graphed in radian mode. Visually, our answer makes sense on the unit circle because the point associated with 3.0877 degrees in the first quadrant is symmetric to the point associated with 3.0877 radians in the second quadrant. Both points have the same y-coordinate.

14. **50** Let a be the number. Since the arithmetic mean of two positive numbers is always greater than or equal to the geometric mean, $\frac{a+2}{2} - \sqrt{2a} = 16$. Isolate the square root before squaring both sides of the equation. The result is $2\sqrt{2a} = a - 30 \longrightarrow 8a = a^2 - 60a + 900 \longrightarrow$ $a^2 - 68a + 900 = 0 \longrightarrow (a-18)(a-50) = 0 \longrightarrow a = 18$ or $a = 50$. Substituting each of these values into the original equation results in rejecting 18 and accepting 50 as the only solution.

15. **18** The two circles in the middle of the diagram are each connected to six other circles. There is only one circle that is not connected to each of them. The only integers that have just one other consecutive integer in the available group are 1 and 8. Therefore, they must be placed in these two central circles, with 2 and 7 at the appropriate ends. The sum of these four numbers is $2 + 8 + 1 + 7 = 18$.



16. **40** The extended Law of Sines states that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$ where r is the radius of the circle circumscribed about ΔABC. Therefore, $c = AB = 2r \cdot \sin C = 2\left(20(\sqrt{6} - \sqrt{2})\right)\sin 105°$. The $\sin 105° = \sin(60° + 45°) = \sin 60° \cos 45° + \sin 45° \cos 60° = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$. It follows that $AB = 40(\sqrt{6} - \sqrt{2})\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) = 40\left(\frac{6-2}{4}\right) = 40$.

17. **405** There are 3^4 four-digit numbers that contain only odd digits in the given set. There are $4(3^4)$ four-digit numbers that contain 1 even and 3 odd digits. Therefore, there are $81 + 324 = 405$ four-digit numbers that contain more odd digits than even digits.
Alternatively, you can subtract the number of four-digit numbers that have 2 even and 2 odd digits from the number of four-digit numbers and then divide by 2. This results in $\frac{6^4 - 6(3^2 \cdot 3^2)}{2} = 405$. We need to divide the difference by 2 because the number of four-digit numbers that contain 4 odd digits or 3 odd digits and 1 even digit equals the number of four-digit numbers that contain 4 even digits or 3 even digits and 1 odd digit.

18. **6** Most calculators will overflow when calculating the number 8^{8^8}. Therefore, calculate $8^8 = 16777216$. Look for a pattern when raising 8 to integer powers:
8, 64, 512, 4096, 32768,… . The units digit follows the pattern 8, 4, 2, 6, 8, 4, 2, 6,… .
Since 16777216 is divisible by 4, the units digit of $8^{16777216}$ is the same as the units digit of 8^4. Thus, the answer is 6.

19. **18** Let $n-1$ be the smallest positive integer. Then $(n-1)^3 + n^3 + (n+1)^3 = (n+2)^3$. Expand the binomials to get $(n^3 - 3n^2 + 3n - 1) + n^3 + (n^3 + 3n^2 + 3n + 1) =$ $n^3 + 6n^2 + 12n + 8 \longrightarrow 2n^3 - 6n^2 - 6n - 8 = 0 \longrightarrow n^3 - 3n^2 - 3n - 4 = 0$. The only possible integer values for n must be factors of 4, that is, 1, 2, or 4. Test them to find that 4 is the only solution to the cubic equation. Therefore, the numbers are 3, 4, 5, and 6, and their sum is 18.
[Note: $3^3 + 4^3 + 5^3 = 6^3$]

20. **70** Let the missing numbers be w, x, y, and z as shown. Since the row, column, and major diagonal products are equal, we can write the extended equality: $18w = 6x = 54y = 3yz = wz = 12z$. The last two equal quantities tell us that $w = 12$, so the common product is $2 \cdot 9 \cdot 12 = 216$. Thus, $x = \frac{216}{6} = 36$, $y = \frac{216}{54} = 4$, and $z = \frac{216}{12} = 18$, and the required sum is $12 + 36 + 4 + 18 = 70$.

2	9	w
x	6	1
3	y	z