Grade Level 9 - NMT 2015 **Solutions**

1. **1** $\big|2-|2+(-5)|\big| = |2-|-3|\,| = |\,2-3\,| = |-1| = 1$

2. **10** Since the "mouth" is $\frac{1}{6}$ of the circle, the "body" is $\frac{5}{6}$ of the circle. The perimeter of the "monster" is $\frac{5}{6}$ of the circle's circumference plus the lengths of two radii. So, the perimeter is $\left(\frac{5}{6}\right)(2\pi)(1)+2 = \left(\frac{5}{3}\right)(\pi)+2$ and $a+b+c = 5+3+2 = 10.$

3. **28** Let $a=3x,\ b=x,$ and $c=5x.$ $\frac{2a+3b}{4b+3c} = \frac{2(3x)+3(x)}{4(x)+3(5x)} = \frac{9x}{19x} = \frac{9}{19}$. Thus, 9 + 19 = 28.

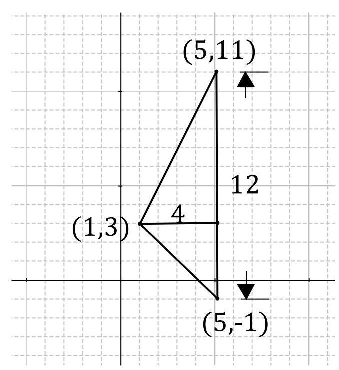
4. **23** Let $x=$ the list price. Aesha's commission is $0.10(x-7)$. Emily's commission is $0.20(x-15)$. Since the commissions are the same, $0.10(x-7) = 0.20(x-15) \to x = 23.$

5. **25** Two intersecting lines form pairs of congruent angles and pairs of supplementary angles. So, $m\angle AED = m\angle BEC$ and $m\angle AED + m\angle AEC = 180$. Thus, $10y-2x = 10x+2y$ and $10y-2x+4y-x = 180 \longrightarrow 8y-12x = 0$ and $14y-3x = 180 \longrightarrow x = 10$ and $y = 15$, so $x+y = 25.$

6. **25** Let $x=$ width and $2x+3=$ length. So, area $=2x^2+3x$ and perimeter $=6x+6$. Thus, $2x^2+3x = 4(6x+6)-13 \longrightarrow 2x^2-21x-11 = 0 \longrightarrow (2x+1)(x-11) = 0 \longrightarrow x = 11$, only. So, the length of the rectangle is 2(11) + 3 = 25.

7. **77** Pairs of the 14 vertices of the polygon can be joined in ${}_{14}C_2 = 91$ ways. Since 14 of these segments are sides of the polygon, there are $91-14 = 77$ diagonals. Note that in general, a convex n-gon has $\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$ diagonals.

8. **24** The given lines determine a triangle whose vertices are $(5,-1)$, $(5,11)$, and $(1,3)$. A base of the triangle is 12 and its corresponding height is 4. Thus, the area is (1/2)(12)(4) = 24.



9. **18** Let x = the number of minutes it takes Justin to get to school.
Kayla's distance is: $\frac{90\text{ steps}}{\text{minute}} \cdot \frac{75\text{ cm}}{\text{step}} \cdot 16$ minutes $= 108000$ cm.
Justin's distance is: $\frac{100\text{ steps}}{\text{minute}} \cdot \frac{60\text{ cm}}{\text{step}} \cdot x$ minutes $= 6000x$ cm.
Since the distances are the same, $6000x = 108000 \longrightarrow x = 18$ minutes.

10. **29** $5^{25}\times 8^{12} = 5^{25}\times 2^{36} = 5^{25}\times 2^{25}\times 2^{11} = 10^{25}\times 2^{11} = 2048\times 10^{25}$. So, 4 digits followed by 25 zeroes is 29 digits.

11. **162** $x^3+y^3 = (x+y)(x^2-xy+y^2) = 6(x^2+y^2-3)$. $(x+y)^2 = x^2+2xy+y^2 = 36 \longrightarrow x^2+y^2 = 36-2(3) \longrightarrow x^2+y^2 = 30$. Thus, $x^3+y^3 = 6(30-3) = 162.$

12. **20** Let $m\angle A = m\angle AFB = x$. By the Exterior Angle and Isosceles Triangle Theorems, $m\angle FBC = m\angle AFB + m\angle A = x + x = 2x = m\angle BCF$. Similarly, $m\angle CFE = x + 2x = 3x = m\angle FEC$, and $m\angle ECD = x + 3x = 4x = m\angle EDC = m\angle AED$. Thus, $x + 4x + 4x = 180$, and $x = m\angle A = 20$.

13. **9** Each factor is a difference of squares. Write each factor as the difference and sum of two terms:
$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\cdots\left(1-\frac{1}{10}\right)\left(1+\frac{1}{10}\right)$.
Collect the factors that are differences and those that are sums:
$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{10}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{10}\right)$.
The difference factors become $\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{8}{9}\right)\left(\frac{9}{10}\right)=\left(\frac{1}{10}\right)$. The sum factors become $\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right)\cdots\left(\frac{10}{9}\right)\left(\frac{11}{10}\right)=\left(\frac{11}{2}\right)$. Therefore, the product is $\left(\frac{1}{10}\right)\cdot\left(\frac{11}{2}\right)=\left(\frac{11}{20}\right)$ and the positive difference between the numerator and denominator is 9.

14. **0** There are 9 digits for numbers 1 through 9, and 180 digits for numbers 10 through 99. The remaining numbers $(2015 - 189 = 1826)$ have 3 digits. Since $\frac{1826}{3} = 608$ r2, we need The 608th number plus 2 digits of the 609th number. The first 3-digit number is 100, and the 608th number is 707. The 609th number is 708, so the 2015th digit is 0, the second digit in 708.

15. **118** There are 6 odd-numbered balls and 5 even-numbered balls in the set. The number of possible sets of 6 balls is ${}_{11}C_6 = 462$. An odd sum must have an odd amount of odd-numbered balls. Thus, 1, 3, or 5 odds must be chosen for the sum to be odd. Choosing 1 odd and 5 evens can happen in ${}_6C_1 \cdot {}_5C_5 = 6$ ways. Choosing 3 odds and 3 evens can happen in ${}_6C_3 \cdot {}_5C_3 = 200$ ways. Choosing 5 odds and 1 even can happen in ${}_6C_5 \cdot {}_5C_1 = 30$ ways. So, an odd sum can happen in 6 + 200 + 30 = 236 ways. Thus, the probability of an odd sum is $\frac{236}{462} = \frac{118}{231}$. So, $a = 118$.

Grade Level 10 - NMT 2015 **Solutions**

1. **124** If x is the original population, then after two years, the population is $(1.4)(1.6)(x) = 2.24x = (1 + 1.24)(x)$. Thus, the population increased by 124%.

2. **3** The prime factorization of 2015 is $5 \cdot 13 \cdot 31$.

3. **2** If x represents both the length of a diagonal of the smaller square and the length of a side of the larger square, then the areas of the squares are $\frac{x^2}{2}$ and x^2, respectively. Thus, the larger square's area is 2 times that of the smaller square.

4. **20** Let the distance $AJ = x$. Since $AD = \frac{3}{8}(AJ)$, $\frac{3}{8}x = -7.5 - (-24) \longrightarrow \frac{3}{8}x = \frac{33}{2} \longrightarrow x = 44$. Thus, the coordinate of point J is $-24 + 44 = 20$.

5. **0** Under a 90° counterclockwise rotation about the origin, the image of $(8, 8)$ is $(-8, 8)$. This point is on the line $y = -x$. So, under a second rotation of 45° counterclockwise, the image of $(-8, 8)$ is on the x-axis with coordinates $\left(-8\sqrt{2}, 0\right)$. Thus, $\left(-8\sqrt{2}\right)(0) = 0$.

6. **7** For $\frac{A}{x+3} + \frac{5}{x-1}$, the common denominator is the product $(x + 3)(x - 1) = x^2 + 2x - 3$ which is the denominator of the given sum. Following the rules of fraction addition, $A(x - 1) + 5(x + 3) = 12x + 8 \longrightarrow Ax - A + 5x + 15 = 12x + 8$. Since $Ax + 5x = 12x$, it follows that $A = 7$. (Note: $-A + 15 = 8$ is also true when $A = 7$).

7. **90** The sum of the measures of the 8 interior angles is $(8 - 2)(180°) = 1080°$ and their average is $135°$. The sum of the measures of the 8 exterior angles is $360°$ and their average is $45°$. Thus, $135° - 45° = 90°$.

8. **60** If x and y are the two numbers, then $xy = 324$ and $\frac{1}{x} = \frac{9}{y}$. So, $y = 9x$, and by substitution, $9x^2 = 324 \longrightarrow x^2 = 36 \longrightarrow x = 6$ and $y = 54$. Thus, $6 + 54 = 60$.

9. **129** The area of the trapezoid is the sum of the areas of the square and the 30°-60°-90° ΔBCE. If $BC = 6\sqrt{3}$, then $CE = 6$, and the area of the trapezoid is $\left(6\sqrt{3}\right)^2 + (0.5)(6)\left(6\sqrt{3}\right) = 108 + 18\sqrt{3}$. Thus, $x + y + z = 108 + 18 + 3 = 129$.
<u>Alternate solution</u>: Using $A = \left(\frac{1}{2}\right)(h)(b_1 + b_2)$, the area of the trapezoid is $\left(\frac{1}{2}\right)\left(6\sqrt{3}\right)\left(6\sqrt{3} + 6\sqrt{3} + 6\right) = \left(3\sqrt{3}\right)\left(12\sqrt{3} + 6\right) = 108 + 18\sqrt{3}$, and $x + y + z = 129$.

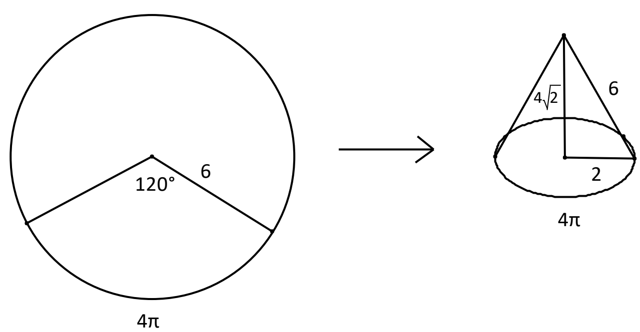
10. **91** Let x be Ricky's present age and $x + 35$ be Robert's present age. Then, $x + 35 + 7 = 5(x - 15) + 5 \longrightarrow x + 42 = 5x - 70 \longrightarrow 4x = 112 \longrightarrow x = 28$ and $x + 35 = 63$. Thus, $28 + 63 = 91$.

11. **220** Let point D be (a, b) so that $ABCD$ is a parallelogram. Since the diagonals of a parallelogram have a common midpoint, $\left(\frac{12+(-7)}{2}, \frac{3+(-1)}{2}\right) = \left(\frac{a+3}{2}, \frac{b+(-3)}{2}\right) \longrightarrow \frac{5}{2} = \frac{a+3}{2}$ and $1 = \frac{b-3}{2} \longrightarrow (a, b) = (2, 5)$. Similarly, if point E is (c, d), then $ABEC$ is a parallelogram and $\left(\frac{12+3}{2}, \frac{3+(-3)}{2}\right) = \left(\frac{c+(-7)}{2}, \frac{d+(-1)}{2}\right) \longrightarrow \frac{15}{2} = \frac{c-7}{2}$ and $0 = \frac{d-1}{2} \longrightarrow (c, d) = (22, 1)$. Thus, $abcd = 2 \cdot 5 \cdot 22 \cdot 1 = 220$.

12. **8** By substitution, $2x^2 + 8x + 11 = 3x + k \longrightarrow 2x^2 + 5x + 11 - k = 0$. In order for this quadratic equation to have two distinct real solutions, the discriminant, $b^2 - 4ac > 0$. So, $5^2 - 4(2)(11 - k) > 0 \longrightarrow 25 - 88 + 8k > 0 \longrightarrow 8k > 63 \longrightarrow k > 7\frac{7}{8}$. Thus, the smallest integral k is 8.

13. **63** The given diagonal has a slope of $\frac{5}{21}$ and a midpoint of $(-6.5, 0.5)$. The other diagonal, which is congruent to, and is the perpendicular bisector of the given diagonal, has the same midpoint as the given diagonal and the negative reciprocal slope of $-\frac{21}{5}$. Using this midpoint and slope, the endpoints of the other diagonal are $(-6.5 - 2.5,\ 0.5 + 10.5)$ and $(-6.5 + 2.5,\ 0.5 - 10.5)$, or $(-9, 11)$ and $(-4, -10)$. Since $(-4, -10)$ is in the third quadrant, the midpoint of the side of the square joining $(-4, -10)$ and $(-17, -2)$ is $\left(-\frac{21}{2}, -6\right)$ and the required product is $\left(-\frac{21}{2}\right)(-6) = 63$.

14. **6** The circumference of the given circle is $2\pi(6) = 12\pi$, and the length of the arc of a sector is $\frac{120}{360}(12\pi) = 4\pi$. When the cone is created, the length of the given circle's radius is the cone's slant height and the sector's arc length is the circumference of the cone's base. So, the length of the radius of the cone is 2, and by the Pythagorean Theorem, the length of the height of the cone is $\sqrt{6^2 - 2^2} = \sqrt{32} = 4\sqrt{2}$. Thus, 4 + 2 = 6.



15. **32** If n represents an outcome of a roll of the weighted number cube, and $P(n)$ is the probability of that outcome, then the probability distribution is given in the table where the prime outcomes are twice as likely to occur. Since the sum of the probabilities is 1, $x = 1/9$.

n	1	2	3	4	5	6
$P(n)$	x	$2x$	$2x$	x	$2x$	x
$P(n)$	1/9	2/9	2/9	1/9	2/9	1/9

So, P(same number on 2 rolls) = $\left(\frac{1}{9}\right)^2 + \left(\frac{2}{9}\right)^2 + \left(\frac{2}{9}\right)^2 + \left(\frac{1}{9}\right)^2 + \left(\frac{2}{9}\right)^2 + \left(\frac{1}{9}\right)^2 = \frac{15}{81} = \frac{5}{27}$. Thus, 5 + 27 = 32.

Grade Level 11 - NMT 2015 **Solutions**

1. **19** $g\left(f(24)\right) = g(-4) = 19$.

2. **43** $\frac{x+9}{x-3} \le 5 \longrightarrow \frac{x+9}{x-3} - 5 \le 0 \longrightarrow \frac{x+9-5x+15}{x-3} \le 0 \longrightarrow \frac{-4x+24}{x-3} = -4\left(\frac{x-6}{x-3}\right) \le 0 \longrightarrow \frac{x-6}{x-3} \ge 0$. Test the intervals on the number line determined by the boundary points $x = 3$ and $x = 6$. Since it is given that $1 \le x \le 10$, the solution set is $[1,3) \vee [6,10]$. The set of integral solutions is $\{1, 2, 6, 7, 8, 9, 10\}$. The required sum is 43.

3. **450** Use the fact that this is a combinations problem involving the counting principle to get ${}_{10}C_2 \cdot {}_5C_2 = 45 \cdot 10 = 450$.

4. **5** $dx^2 = k \longrightarrow 100(4)^2 = k \longrightarrow k = 1600 \longrightarrow 64x^2 = 1600 \longrightarrow x^2 = 25 \longrightarrow x = 5$.

5. **34** $\frac{2}{9-\sqrt{13}} = \frac{2(9+\sqrt{13})}{68} = \frac{9+\sqrt{13}}{34}$. The denominator of the simplified result is 34.

6. **69** Use the theorem that the sum of the lengths of any two sides of a triangle must be greater than the length of the third side. If the length of the third side is x, then $x < 35$. So, $x = 34$ and the maximum possible perimeter is $15 + 20 + 34 = 69$.

7. **7** The slope of the given line is $\frac{10-(-5)}{-3-2} = \frac{15}{-5} = -3$. So, the slope of the perpendicular line is $\frac{1}{3}$, and the equation of the perpendicular line is $y - 5 = \frac{1}{3}(x + 6)$ or $y = \frac{1}{3}x + 7$. Thus, the y-intercept is 7.

8. **385** Use the law of the logarithm of a product to re-write the given equation as $\log_b(5 \cdot 7 \cdot 11) = 1 \longrightarrow b = 5 \cdot 7 \cdot 11 = 385$.

9. **9** Multiply by 5^x to get $5^{2x} + 5^3 = 30 \cdot 5^x \longrightarrow 5^{2x} - 30 \cdot 5^x + 125 = 0 \longrightarrow (5^x - 25)(5^x - 5) = 0 \longrightarrow x = 2$ or $x = 1$. The sum of the cubes of the roots is 9.

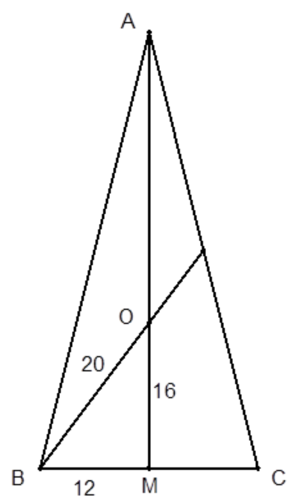
10. **8** $y = |2x - 5| - 4$ has a vertex at $\left(\frac{5}{2}, -4\right)$ and intersects the x-axis at $\left(\frac{1}{2}, 0\right)$ and $\left(\frac{9}{2}, 0\right)$. The region is a triangle whose base and height are each 4. Its area is 8.

11. **684** Use the sequence definition to get $342, 684, 171, 342, 684, 171, \cdots$. Notice that any a_n where n is a multiple of 3 is 171. Since $2015 \equiv 2 \bmod 3$, (leaves a remainder of 2 upon division by 3), $a_{2015} = a_2 = 684$.

12. **14** Name points M and G to be the midpoints of $\overline{DE}$ and $\overline{DF}$ respectively. The length of $\overline{OB}$, a diagonal of the square is 4. Hence, the length of each side of the square is $2\sqrt{2}$. The diagonals of a square are congruent and perpendicular bisectors of each other. Therefore, the coordinates of points A and C are $(2,2)$ and $(-2,2)$ respectively. The length of $\overline{EG}$ is 4 and because ΔEDG is $30° - 60° - 90°$, $DG = \frac{4}{\sqrt{3}}$. So, the x-coordinate of point E is $4 + \frac{4}{\sqrt{3}}$ and the coordinates of point M are $\left(4 + \frac{2}{\sqrt{3}}, 2\right)$.

So, $AM = 4 + \frac{2}{\sqrt{3}} - 2 = 2 + \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}+2}{\sqrt{3}} = \frac{6+2\sqrt{3}}{3}$. The required sum is 6+2+3+3=14.

13. **576** The triangle is isosceles and the medians meet at the centroid, point O, which is 2/3 of the way from any vertex to the midpoint of the opposite side. The longest median of length 48 is the perpendicular bisector of the base. Use the accompanying figure. Since $OM = 16$ and $OB = 20$, then $MB = 12$ and $BC = 24$. Then using absolute value for area,
$|ABC| = \frac{1}{2} BC \cdot AM = \frac{1}{2} \cdot 24 \cdot 48 = 576.$



14. **180** It is given that $(\cos x)^4 + (\sin x)^4 = \frac{3}{4}$. Start with $(\cos^2 x + \sin^2 x)^2 = \cos^4 x + \sin^4 x + 2\cos^2 x \sin^2 x \longrightarrow 1 = \frac{3}{4} + \frac{1}{2}\sin^2 2x \longrightarrow \sin^2 2x = \frac{1}{2} \longrightarrow \sin 2x = \pm\frac{\sqrt{2}}{2}$.
So, $2x = 45°, 135°, 225°, 315°, 405°, 495°, 585°, 675°$
and $x = 22.5°, 67.5°, 112.5°, 157.5°, 202.5°, 247.5°, 292.5° 337.5°$
Notice that there are four pairs of angles (first and eighth, second and seventh, etc.), the sum of whose measures is 360. Thus, the required average is 180.

15. **8** Solve simultaneously by substitution: $x^2 + (x+b)^2 = 4 \longrightarrow 2x^2 + 2bx + b^2 - 4 = 0$.
Since there is only one solution, set the discriminant to zero:
$(2b)^2 - 4 \cdot 2(b^2 - 4) = 0 \longrightarrow -4b^2 = -32 \longrightarrow b^2 = 8.$

Grade Level 12 - NMT 2015 **Solutions**

1. **42** Matrices can be used to solve the system of equations, but it is much easier to add all 3 equations, which yields $6x + 6y + 6z = 252$. Dividing both sides by 6 gives us $x + y + z = 42$.

2. **128** There are numerous algebraic methods, but one of the most straightforward is to convert the equation to exponential form: $8x^2 = 2^{17}$. Since $8 = 2^3$, $x^2 = 2^{14}$. Thus, $x = 2^7 = 128$.

3. **7** Using permutations with repetition, the number of permutations is $\frac{8!}{2!2!2!} = \frac{8!}{8} = \frac{8\cdot 7!}{8} = 7!$. So, $n = 7$.

4. **170** Method 1: Multiply both sides by $x^{1/2}$ to get $x^2 + 77 = 18x$. Factoring and solving the quadratic produces $x = \{7, 11\}$. The sum of the squares of the roots is $7^2 + 11^2 = 170$. Method 2: If r_1 and r_2 are the roots, and using the sum and product of the roots of a quadratic equation, $r_1^2 + r_2^2 = (r_1 + r_2)^2 - 2r_1r_2 = 18^2 - 2 \cdot 77 = 324 - 154 = 170$.

5. **864** Method 1: Using the chain rule: $f'(x) = e^{4\ln x} \cdot \frac{4}{x}$, so $f'(6) = e^{4\ln 6} \cdot \frac{4}{6} = e^{\ln(6^4)} \cdot \frac{2}{3} = 6^4 \cdot \frac{2}{3} = 1296 \cdot \frac{2}{3} = 864$.
Method 2: Simplify the original function: $f(x) = e^{\ln(x^4)} = x^4$, so $f'(x) = 4x^3$, and $f'(6) = 4 \cdot 6^3 = 4 \cdot 216 = 864$.

6. **972** Let $m\angle CAD = m\angle DAB = x$. Since ΔADB is isosceles, $m\angle ABD = x$ as well. Using the sum of the measures of the angles of ΔABC, $90° + 3x = 180°$, so $x = 30°$. This means that ΔABC is a $30°$-$60°$-$90°$ triangle. With $AC = 6$, we know that $BC = 6\sqrt{3}$ and so the area of ΔABC is $18\sqrt{3}$. The square of the area is $324 \cdot 3 = 972$.

7. **995** $i^{2!} = i^2 = -1$. $i^{3!} = i^6 = -1$. For all $k \geq 4$, $k!$ is a multiple of 4, so for all of these values, $i^{k!} = 1$. This means that $\sum_{k=2}^{1000} i^{k!} = -1 - 1 + 1 + 1 + 1 + \cdots + 1 = -1(2) + 1(997) = 995$.

8. **50** The given equation is equivalent to $x^2 + 5x = \pm 6$. Solving both equations by factoring gives four solutions: $-6, 1, -3$, and -2. Summing the squares of these four solutions yields $36 + 1 + 9 + 4 = 50$.

9. **196** If the dimensions are given by l, w, h, then $lwh = 2015$. Since the total edge length is given by $4l + 4w + 4h$, we want to minimize $l + w + h$. The prime factorization of 2015 is $5 \cdot 13 \cdot 31$, so we can have $l + w + h = 49$. Any other factorization of 2015 must include a number greater than or equal to 65 (why?), so the minimum total edge length is $4(5 + 13 + 31) = 196$.

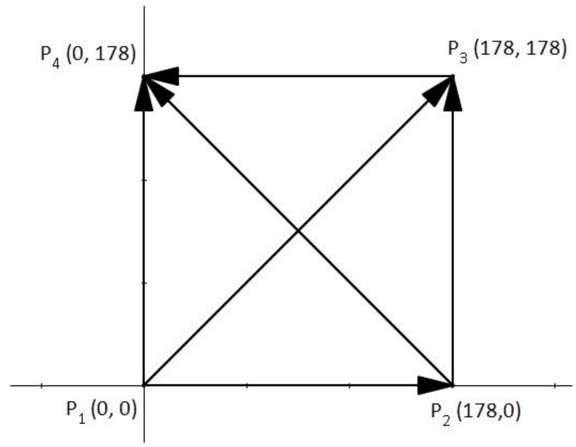
10. **0** This one is a time-killer if you try to solve the related rates problem as given. Notice that $\angle APB$ is inscribed in a semicircle, so $m\angle APB$ is a constant $90°$. Therefore, $\frac{d\theta}{dt} = 0$.

11. **13** Factor $f(x)=\frac{12x(x-6)(x-1)}{5(x-6)(x-1)}$, which has removable discontinuities at $x=1$ and $x=6$. Plugging these values of x into the reduced fraction $\frac{12x}{5}$ yields holes at $\left(1,\frac{12}{5}\right)$ and $\left(6,\frac{72}{5}\right)$. Applying the distance formula to these two points produces
$\sqrt{(6-1)^2+\left(\frac{72}{5}-\frac{12}{5}\right)^2}=\sqrt{25+144}=13.$

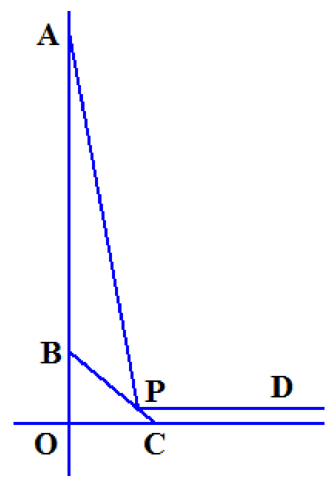
12. **888** In order for $g(a)$ to be zero, $g(1776-a)$ must be zero as well. Since there is only one value of a for which this is true, $a=1776-a\longrightarrow 2a=1776\longrightarrow a=888$.

13. **625** $g(x)=f\big(f(x)\big)$. Using the Chain Rule, $g'(x)=f'\big(f(x)\big)\cdot f'(x)$. So,
$g'(5)=f'\big(f(5)\big)\cdot f'(5)=f'(5)\cdot f'(5)=[f'(5)]^2=[5\cdot f(5)]^2=[5\cdot 5]^2=625.$

14. **712** If the perimeter of the square is 712, the length of each side is 178. The square is graphed with its four vertices labeled as shown in the diagram. Using the fact that if point $A(x_1,y_1)$, and point $B(x_2,y_2)$ determine a vector $\overrightarrow{AB}$ with components $\langle x_2-x_1,\ y_2-y_1\rangle$, then the vectors representing the sides and diagonals of the square are as follows: $\overrightarrow{P_1P_2}=\langle 178,0\rangle$, $\overrightarrow{P_1P_3}=\langle 178,178\rangle$, $\overrightarrow{P_1P_4}=\langle 0,178\rangle$, $\overrightarrow{P_2P_3}=\langle 0,178\rangle$, $\overrightarrow{P_2P_4}=\langle -178,178\rangle$, and $\overrightarrow{P_3P_4}=\langle -178,0\rangle$. Thus, the sum of the vectors $\vec{v}=\langle 0,\ 4(178)\rangle$ and the magnitude of $\vec{v}=\sqrt{0+4^2(178)^2}=4(178)=712$. Note that this is also the perimeter of the square.



15. **121** Let $m\angle PCO=\theta$. By alternate interior angles, $m\angle CPD=\theta$ and by reflection, $m\angle BPA=\theta$ as well. We know that $m\angle OBC=90°-\theta$ and so $m\angle ABP=90°+\theta$, leading to $m\angle BAP=90°-2\theta$. We want $\cos(90°-2\theta)$ which equals $\sin 2\theta$ by a cofunction identity. $\sin 2\theta=2\sin\theta\cos\theta$. Using right ΔBOC with x-intercept 6 and y-intercept 5, $\sin\theta=\frac{5}{\sqrt{61}}$ and $\cos\theta=\frac{6}{\sqrt{61}}$. Therefore, $\cos(\angle BAP)=2\cdot\frac{5}{\sqrt{61}}\cdot\frac{6}{\sqrt{61}}=\frac{60}{61}$. The answer is $60+61=121$.

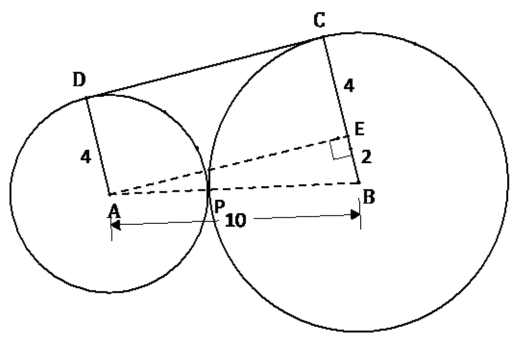


1. **17** If $n = 3x - 31$, then $n^2 - 17n - 60 = 0 \longrightarrow (n-20)(n+3) = 0 \longrightarrow n = 20$.
Thus, $3x - 31 = 20 \longrightarrow x = 17$.

2. **156** The lengths of the diameter and radius increase by a factor of 1.6, so the area increases by a factor of $(1.6)^2 = 2.56 = (1 + 1.56)$. Thus, the rate of increase is 1.56 = 156%. So, n = 156.

3. **498** For the three numbers, the largest median occurs when $x = 1$ and $z = y + 1$. Since the sum is less than 1000, $1 + y + y + 1 < 1000 \longrightarrow y < 499$. Thus, the largest y is 498.

4. **96** From point A, draw $\overline{AE}$ so that point E is on $\overline{BC}$ and $\overline{AE} \perp \overline{BC}$. This forms rectangle $AECD$ with $\overline{DC} \cong \overline{AE}$. Also, draw $\overline{AB}$ which passes through P, the two circles' point of tangency, forming right ΔAEB. Since $AB = AP + PB = 4 + 6 = 10^*$, and $EB = 2$, then by the Pythagorean Theorem, $(AE)^2 = 10^2 - 2^2 = 96$. Thus, $(DC)^2 = 96$.
*Note to the reader: We are asserting that points *A, P,* and *B* are collinear. Can you prove it?



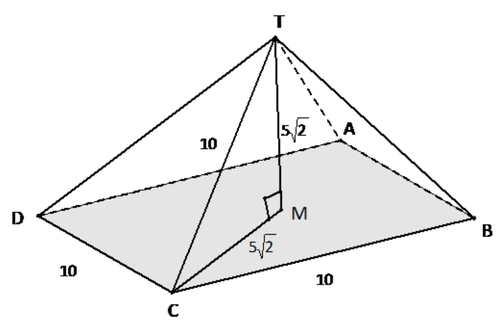
5. **936** Since there are 26 uppercase letters, 26 lowercase letters, and 10 digits,
$N = 26^6 \cdot 10^2$ and $M = 52^6$. Thus, $\frac{(N-M)}{26^5} = \frac{26^6(10^2) - 26^6(2^6)}{26^5} = \frac{26^6(100-64)}{26^5} = \frac{26^6(36)}{26^5} = (26)(36) = 936$.

6. **170** The common difference between consecutive terms, $d = \frac{a_{17} - a_{11}}{17-11} = \frac{20.15 - 19.7}{6} = 0.075$. Since the n^{th} term, a_n, of an arithmetic sequence is defined by $a_n = a_1 + d(n-1)$, the first term, $a_1 = a_{11} - (0.075)(11-1) = 18.95$. So, the given sequence is represented by the formula $a_n = 18.95 + (0.075)(n-1)$. Thus, $a_{2015} = 170$.

7. **310** Solve $x = 3y + 2$ to get the inverse of the given function, $f^{-1}(x) = \frac{x-2}{3}$. Apply f^{-1} to both sides of $f(f(f(t))) = 929$ to get $f(f(t)) = f^{-1}(929) = 309$. Apply f^{-1} a second time to get $f(t) = f^{-1}(309) = \frac{307}{3}$. Finally, we apply f^{-1} a third time to get $t = f^{-1}\left(\frac{307}{3}\right) = \frac{301}{9}$. Thus, $a + b = 310$.

8. **281** There are $6! = 720$ words in the sequence; $5! = 120$ beginning with E, 120 beginning with I, and so on. We conclude that the 241st term is the first word in the sequence beginning with L: **LEINST.** There are $4! = 24$ words beginning with LE, so the 265th word is **LIENST.** Now there are $3! = 6$ words beginning with LIE, and 6 words with LIN, so the 277th word is **LISENT.** Counting forward from there, we have **LISETN, LISNET, LISNTE,** and the 281st word, **LISTEN.**

9. **236** The volume of a pyramid is given by $V = \frac{1}{3}Bh$ where B is the base area and h is the height. In the diagram, point M is the center of the square base, and $\overline{TM}$, which is perpendicular to the base is the height of the pyramid. A diagonal of the base measures $10\sqrt{2}$, and ΔTMC is a right triangle with $TC = 10$ and $MC = 5\sqrt{2}$. So, by the Pythagorean Theorem (or by recognizing that ΔTMC is an isosceles right triangle), $TM = 5\sqrt{2}$ and the volume of the pyramid is $\frac{1}{3}(100)(5\sqrt{2}) \approx 236$.



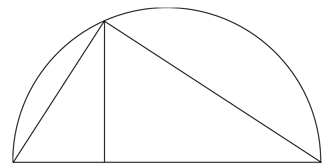
10. **429** If we rationalize the denominators, we get a telescoping series, that is, a series in which all but two terms add to zero, or which "collapses" like a telescope:
$N = \sqrt{5} - 2 + \sqrt{6} - \sqrt{5} + \sqrt{7} - \sqrt{6} + \cdots + \sqrt{2015} - \sqrt{2014}$. The remaining terms of N are $\sqrt{2015} - 2$. So, $10N = 10\sqrt{2015} - 20 \approx 429$.

Team Problem Solving - NMT 2015 Solutions

1. **34** Since the sum of two consecutive terms is the next term in the sequence, the ninth term added to 8 must be 10. Therefore, the ninth term is 2. The seventh term is $2 - 8 = -6$ and the sixth term is $8 - (-6) = 14$. The fifth term is $-6 - 14 = -20$ and the fourth term is $14 - (-20) = 34$.

2. **20** Let $y = \sqrt{x}$; so, $x = y^2$. Then solve $y^2 - 6y + 8 = 0 \longrightarrow (y-4)(y-2) = 0 \longrightarrow y = 4$ or $y = 2$. Replace x in the solutions to get $x = 16$ or $x = 4$. So, the sum of the roots is $16 + 4 = 20$. [Note: The sum of the roots can be found without finding either of the roots. The sum of the roots of the equation $y^2 - 6y + 8 = 0$ is $-(-6) = 6$ and the product of the roots is 8. Since $x_1 + x_2 = {y_1}^2 + {y_2}^2$ and $(y_1 + y_2)^2 = {y_1}^2 + 2y_1 y_2 + {y_2}^2$, $x_1 + x_2 = {y_1}^2 + {y_2}^2 = (y_1 + y_2)^2 - 2y_1 y_2 = 6^2 - 2(8) = 36 - 16 = 20$.]

3. **50** Since $a^2 + b^2 + c^2 = 400$, and $a^2 + b^2 = c^2$, $2c^2 = 400 \longrightarrow c^2 = 200$. If we examine the right triangle in a semicircle with the length of the hypotenuse fixed, we can see that the maximum distance to the hypotenuse from the vertex of the right angle is the length of a radius. Hence, the maximum area occurs when the legs are equal. Thus, $a = b$ and the area of the triangle is $\frac{1}{2}ab = \frac{1}{2}a^2 = \frac{1}{2}(100) = 50$.



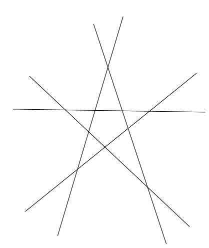
4. **123** Let the number be $10t + u$ where t and u are the digits of the 2-digit number. Then $(10t + u) - tu = 14$. Solve for t: $t = \frac{14-u}{10-u}$ and create a table for all possible values for u and t. The only integer values for t form the numbers 26, 38, and 59. The sum of these is 123.

u	0	1	2	3	4	5	6	7	8	9
t	1.4	1.44	1.5	1.57	1.67	1.8	2	2.33	3	5

5. **31** We know that $p - r = q^2 - s^4 = 73 \longrightarrow (q - s^2)(q + s^2) = 73$. Since the only integer divisors of 73 are 1 and 73, $q - s^2 = 1$ and $q + s^2 = 73$. Add these two equations together to get $2q = 74 \longrightarrow q = 37$ and $s = 6$. Thus, $q - s = 31$.

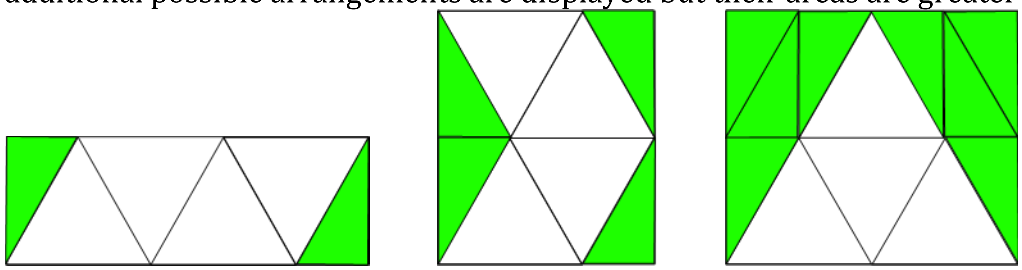
6. **9** Multiply the two expressions on the left and combine like terms. The result is $\sqrt{x^2} = x$. This is true for all $x \geq 0$. However, the domain for $\sqrt{81 - x^2}$ is $-9 \leq x \leq 9$, so the solution to the given equation is $0 \leq x \leq 9$ and the sum of the two extremes is $0 + 9 = 9$.

7. **16** In order to maximize the number of regions, draw the lines so that no two are parallel and no three are concurrent. The ordered pairs (lines, regions) are: (1, 2), (2, 4), (3, 7), (4, 11), and (5, 16).



8. **729** To eliminate the outer absolute value symbol, set up the equivalent sentence $6 - |x| = 3$ or $6 - |x| = -3$. Solve the first to get $x = \pm 3$ and the second to get $x = \pm 9$. The product of the four solutions is $(3)(-3)(9)(-9) = 729$.

9. **85** The arrangement of the four equilateral triangles determines the dimensions of the rectangle that contains them. The arrangement that allows the rectangle to be as small as possible is the linear arrangement shown. The area of the rectangle is 5 times the area of the triangle since the two right triangles on the ends can combine to form a fifth equilateral triangle. Two additional possible arrangements are displayed but their areas are greater than $5(17) = 85$.



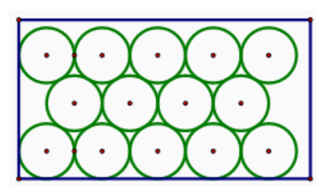
10. **60** Since the product of the two sides of the triangle not given is 255, we need to consider the factors of 255. The list of all positive factors is 1, 3, 5, 15, 17, 51, 85, and 255. We know that the sum of the lengths of the two shorter sides of a triangle must be greater than the length of the third side. Therefore, if we let side $a = 8$ and assume $b < c$ then lengths 1 and 255, 3 and 85, and 5 and 51 are not possible, since in each case the sum of the two shorter sides is not greater than the third side. This leaves 15 and 17 as the other two sides. A triangle whose sides have lengths 8, 15, and 17 is a right triangle since $8^2 + 15^2 = 17^2$. So the area of the triangle is $\frac{(8)(15)}{2} = 60$.

11. **1** Solve for all four solutions: $(\log_3 x^2)^2 = 9 \longrightarrow \log_3 x^2 = \pm 3 \longrightarrow x^2 = 3^3 = 27 \longrightarrow$ $x = \pm\sqrt{27}$ or $x^2 = 3^{-3} = \frac{1}{27} \longrightarrow x = \pm\frac{1}{\sqrt{27}}$. The product is $\left(\sqrt{27}\right)\left(\frac{1}{\sqrt{27}}\right)\left(-\sqrt{27}\right)\left(-\frac{1}{\sqrt{27}}\right) = 1$.

12. **6** The x- and y-intercepts are equal when the slope of the line is -1. Place the equation into slope-intercept form and find k. Substitute to find the intercepts. $6y - 8ky = -4kx - 3x + 20k + 7 \longrightarrow y = \frac{-4k-3}{6-8k}x + \frac{20k+7}{6-8k} \longrightarrow \frac{-4k-3}{6-8k} = -1 \longrightarrow k = \frac{1}{4}$. The y–intercept is $\frac{20(1/4)+7}{6-8(1/4)} = \frac{12}{4} = 3$. Since both intercepts are the same, the sum is $2(3) = 6$.

13. **162** Since the perpendicular bisector of one leg of the triangle also contains the midpoint of the base, it must be parallel to the other leg of the triangle. Therefore, the isosceles triangle must also be a right triangle whose area is $(1/2)(18)(18) = 162$.

14. **14** The diagram shows the maximum number of circles that can pack into the rectangle. Since the height is only 5.5, the circles do not stack 3 high if placed one directly above the other. The height as stacked in the diagram is $1 + 2\sqrt{3} + 1 \approx 5.464$. There is not enough room to fit another circle in the middle row. Thus, the maximum number of circles that can pack into the rectangle is $5 + 4 + 5 = 14$.



15. **15** Let the slope be m and the y-intercept be b. Then, $m + b = \frac{47}{6}$ and $\frac{1}{m} + b = \frac{32}{9}$. Subtract the second equation from the first to eliminate b. The result is $m - \frac{1}{m} = \frac{77}{18}$. Therefore, $18m^2 - 77m - 18 = 0 \longrightarrow (9m + 2)(2m - 9) = 0$. Since the slope is positive, it must be $\frac{9}{2}$. Substitute to find $b = \frac{10}{3}$. Then, $\left(\frac{9}{2}\right) \cdot \left(\frac{10}{3}\right) = 15$.

16. **15** A number is divisible by 33 when it is divisible by both 3 and 11. A number is divisible by 11 when the sum of the alternating digits minus the sum of the remaining digits is a multiple of 11. That is, $(4+8+d)-(c+d+c)=$ a multiple of $11 \longrightarrow 12-2c=11k$ for some integer k. When $k=0$, $c=6$ and no other value of k produces a single-digit solution. The sum of the digits in the original number must be a multiple of 3. That is, $12+2c+2d=3j$ for some integer j. Since $c=6$, we need d to be a multiple of 3. The only single-digit multiple of 3 greater than 6 is 9, so $d=9$ and $c+d=15$.

17. **512** Method 1: $T_7 = 1+2+3+\cdots+7=\frac{7}{2}(1+7)=28$, so $T_8=28+8=36$. Now we can calculate the difference of the squares: $36^2-28^2=(36-28)(36+28)=8(64)=512$. This generalizes to ${T_n}^2-{T_{n-1}}^2=(T_n-T_{n-1})(T_n+T_{n-1})=n(2T_{n-1}+n)$. Method 2: Learn the formula ${T_n}^2-{T_{n-1}}^2=n^3$. This may be derived from $T_n=\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$ and is left to the reader.

18. **525** Since $\log_3 \tan y = -\frac{1}{2} \longrightarrow \tan y = 3^{-1/2}=\frac{1}{\sqrt{3}} \longrightarrow y=30°+180°k$, where k is an integer. Since y has the specified domain $[180°, 270°]$, $y=210°$. Since $\log_2 \cos x=-\frac{1}{2} \longrightarrow \cos x = 2^{-1/2}=\frac{1}{\sqrt{2}} \longrightarrow x=45°+360°k$ or $315°+360°k$, where k is an integer. Since x has the specified domain $[270°, 360°]$, $x=315°$. Thus, the sum of the degree values for x and y is $210+315=525$.

19. **120** Let the three positive integers be $n-1$, n, and $n+1$. Then, $(n-1)(n)(n+1)=24\left(\frac{3n}{3}\right) \longrightarrow n^3-n=24n \longrightarrow n^2=25 \longrightarrow n=5$. The product, $4\times5\times6=120$.

20. **31** One place to start is 2 down. There are three palindromic squares of three-digits to consider: 121, 484, and 676. Now make a list of Fibonacci numbers to include 4-digit numbers. There are four of these: 1597, 2584, 4181, and 6765. So, now we can conclude that 1 across and 2 down must be 4181 and 121, respectively. Next consider 1 down, a Pythagorean triple. It must be 4-3-5 or 4-5-3. Since 6 across consists of four consecutive numbers, it must be 4-5-3. Consider 5 across which must be a multiple of 401. Divide 5200 by 401 to get 12.96..., so multiply 401 by 13 to get 5213. Now 3 down must be 8-1-0, and 4 down must be 1-3-2. So, the sum of all twelve digits is 31.

1 **4**	2 **1**	3 **8**	4 **1**
5 **5**	**2**	**1**	**3**
6 **3**	**1**	**0**	**2**

Tie Breakers - NMT 2015 **Solutions**

1.
2.
3.