Grade Level 9 - NMT 2014 **Solutions**

1. **29** Find the mean (average) of the two numbers by dividing their sum by 2. Since 1/6 + 1/4 = 10/24 = 5/12 and one-half of that is 5/24, the sum of 5 and 24 = 29.

2. **9** Find the largest perfect cube factor of 320. Since 320 = 64 × 5 we get $\sqrt[3]{320} = \sqrt[3]{64} \cdot \sqrt[3]{5} = 4\sqrt[3]{5}$. Therefore $a + b = 4 + 5 = 9$.

3. **10** Each of Susan's tickets cost .75($20) = $15 and each of Partha's tickets cost .70($20) = $14. The difference in their payments is 5(14) – 4(15) = 70 – 60 = 10.

4. **23** There are ${}_9C_3 = 84$ ways to choose 3 points given 9 points. Since all the rows, columns, and 2 diagonals form a straight line, 8 of the 84 ways form straight lines. The probability of picking one of these 8 is 8/84 = 2/21 = a/b and $a + b = 2 + 21 = 23$.

5. **8** Follow the steps that the teacher gave: 6 + 3 = 9 → 9 × 2 = 18 → 18 – 5 = 13. Pass 13 to Sibyl. 13 – 1 = 12 → (1/3) × 12 = 4 → 2 × 4 = 8.

6. **50** Since the area of each L-shaped region is 3/16 of the area of the large outer square, the area of the small inner square is 1 – 4(3/16) = 4/16 = 1/4 of the area of the larger square. A square with area 1/4 of another square has a side that is 1/2 of the larger square and (1/2)(100) = 50.

7. **0** Multiply $(p-1)(q-1) = pq - p - q + 1 = pq - (p+q) + 1$. The product of the 2 roots (pq) in the equation $2x^2 + 3x - 5 = 0$ is $c/a = -5/2$ and the sum $(p + q)$ is $-b/a = -3/2$. Therefore the value of $(p - 1)(q - 1) = -5/2 - (-3/2) + 1 = -2/2 + 1 = 0$.
Alternate solution: Solve the given equation for p and q. $2x^2 + 3x - 5 = 0 \rightarrow (2x + 5)(x - 1) = 0$ so the roots are –5/2 and 1. Subtract 1 from each root. Since one number becomes 0, the product is 0.

8. **7** Average rate = total distance/ total time. Apply $d = rt$ to calculate the distance for each part of the trip. The distance traveled on the bicycle = 16(1/2) = 8 miles and the distance walking = 4(3/2) = 6 miles. Thus the average rate = (8 + 6)/(1/2 + 3/2) = 14/2 = 7 mph.

9. **74** The prime factors of 2014 are 2, 19, and 53. The sum 2 + 19 + 53 is 74.

10. **9** The units digits of the powers of 43 are the same as the units digits of powers of 3: $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$. The pattern continues as 3, 9, 7, 1, 3, 9, 7, 1, 3, … . Divide 2014 by 4 to find the remainder is 2. Therefore the units digit for 43^{2014} is the same as 43^2 or $3^2 = 9$.

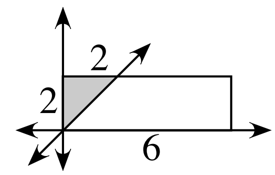
11. **6** The area of ΔABE = (.5)(10)(4) = 20 and the area of ΔBAC = (.5)(7)(4) = 14. By subtracting the area of their common overlapping region, ΔADC, from each of these values, the difference remains the same. Thus the difference in the areas of ΔADE and ΔBDC is 20 – 14 = 6.

12. **50** Rather than find each of the sums, realize that the difference of each even and the previous odd is 1. There are 50 such pairs of odd and even numbers so the sum of all of these pairs is 50.

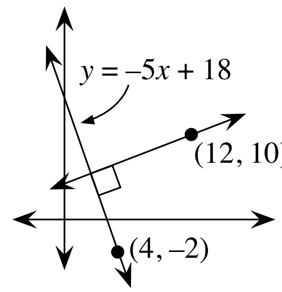
13. **2** Factor $x^2 - y^2 = (x-y)(x+y) = 140$. Consider all of the factor pairs of 140: (1, 140), (2, 70), (4, 35), (5, 28), (7, 20), and (10, 14). When the difference of two integers is even, so is the sum and when the difference of two integers is odd so is the sum. Therefore the only possible pairs of factors that would produce an x and y are (2, 70) and (10, 14) so the answer is 2. To find the values of x and y

solve the system of equation for each pair. Example: $x - y = 2$ and $x + y = 70$. Add these equations to get $2x = 72$ so $x = 36$ and $y = 34$.

14. **5** Sketch the graph of the rectangle and the line $y = x$. The area of the shaded triangle is the region that has $x < y$. Therefore the probability that a point chosen from inside the rectangle will be inside the triangle is determined by dividing the area of the triangle by the area of the rectangle = 2/12 = 1/6. Since $a = 1$ and $b = 6$, $|a-b| = |1-6| = 5$.



15. **10** The point (12, 10) is closest to the line $y = -5x + 18$ at the point where the line perpendicular to the given line, containing the given point, intersects the given line. To find this point, first find the equation of the perpendicular line. It has a slope that is the negative reciprocal of –5 or 1/5. The equation is $y - 10 = (1/5)(x - 12)$ or $y = (x + 38)/5$. Set the two expressions for y equal to each other and solve: $-5x + 18 = (x + 38)/5 \rightarrow -25x + 90 = x + 38 \rightarrow 26x = 52 \rightarrow x = 2$, Substitute to get $y = 8$ and then find the sum of these to answer the question: 2 + 8 = 10.



Grade Level 10 - NMT 2014 **Solutions**

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Grade Level 11 - NMT 2014 **Solutions**

1. **10** The average number of honors classes per student is 3, so $\frac{0\cdot 2+1\cdot 4+2\cdot 4+3\cdot 6+4x+5\cdot 4}{20+x}=3$.

   Simplify and solve: $\frac{50+4x}{20+x}=3\rightarrow 50+4x=60+3x\rightarrow x=10$.

2. **180** Since there are 6 letters in the word KANSAS with two A's and 2 S's there are $\frac{6!}{2!2!}=180$.

3. **14** Since $|4a+6|+2a=18$ either $4a+6=18-2a$ or $-(4a+6)=18-2a$. Therefore either $a=2$ or -12 and $|2-(-12)|=14$.

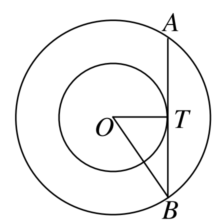
4. **741** The number of seats in successive rows forms an arithmetic sequence. The n^{th} term is $a_n=a_1+(n-1)d$ and the sum of the n terms is found by $S_n=\frac{n}{2}(a_1+a_n)$. Therefore

   $a_n=12+(19-1)\cdot 3=66$ and $S_n=\frac{19}{2}(12+66)=741$.

5. **23** Use completing the squares to convert the equation of the circle from general form to standard form: $x^2-6x+y^2+8y-504=0\rightarrow x^2-6x+9+y^2+8y+16=504+9+16=(x-3)^2+(y+4)^2=23^2$. From this form we find the center coordinates to be $(3,-4)$ and the radius is 23

6. **329** To be successful, the first roll can be any of the six numbers. The second roll can be any of the five remaining numbers, and so on until one number remains. So the requested probability can be found using $\frac{6}{6}\cdot\frac{5}{6}\cdot\frac{4}{6}\cdot\frac{3}{6}\cdot\frac{2}{6}\cdot\frac{1}{6}=\frac{5}{324}$ and $5+324=329$.

7. **48** Let the common center be point O. Let the endpoints of the chord of the larger circle be A and B, with point of tangency T. Then $OT=10$ and $OB=26$. Since $\overline{OT}$ is a radius and $\overline{ATB}$ is a tangent segment, $AT=TB$ and $\overline{OT}\perp\overline{ATB}$. So $\angle BTO$ is a right angle. By the Pythagorean theorem, $10^2+TB^2=26^2$, so $TB=24$ and $AB=48$.

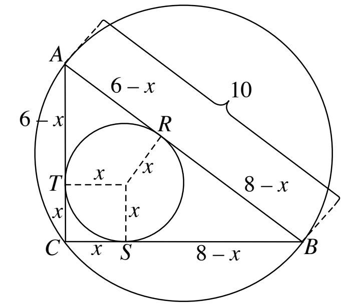


8. **840** The general solution of the given equation, in degrees, is $3x=90+360k$, where k is an integer. It follows that $x=30+120k$ and thus $x=30, 150, 270$, and 390. The sum of these solutions is 840.

9. **147** Apply the definition of logarithm as the inverse of an exponent: $log_3 7=a\rightarrow 3^a=7$ and $log_3 b=2a+1\rightarrow 3^{2a+1}=b$. Therefore $b=\left(3^a\right)^2\cdot 3=7^2\cdot 3=147$.

10. **45** The given equation is equivalent to $4+\frac{1}{x^4}=\frac{1}{x^2}$. After clearing fractions: $4x^4+1=5x^2$ so $4x^4-5x^2+1=0\rightarrow(x^2-1)(4x^2-1)=(x-1)(x+1)(2x-1)(2x+1)=0$. The four roots are ± 1 and $\pm\frac{1}{2}$. Finally, $15\cdot\sum_{k=1}^{4}|r_k|=15\left(1+1+\frac{1}{2}+\frac{1}{2}\right)=15(3)=45$.

11. **21** Based on the converse of the Pythagorean theorem, $\angle C$ is a right angle so ΔABC is a right triangle. The center of the circumscribed circle is the midpoint of the hypotenuse $\overline{AB}$. The radius of the circumscribed circle is 5 and its area is 25π. Let the points of tangency of the inscribed circle on $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ be R, S, and T respectively. Let $TC = x$. Therefore, $CS = x$. Similarly, $AT = AR = 6 - x$ and $BS = BR = 8 - x$. Since $AR + RB = AB$, $6 - x + 8 - x = 10$ and $x = 2$ and the area of the inscribed circle is 4π. Since $k\pi = 25\pi - 4\pi = 21\pi$, $k = 21$.



12. **2** The coordinates of point B are $(8\cos 120°, 8\sin 120°) = \left(-4, 4\sqrt{3}\right)$. Using the midpoint formula, the coordinates of point M are $\left(\dfrac{8+(-4)}{2}, \dfrac{0+4\sqrt{3}}{2}\right) = \left(2, 2\sqrt{3}\right)$. Therefore $x = 2$.

13. **95** Since $3 + 4i$ is a root and the coefficients are real, the conjugate, $3 - 4i$. The sum of these two roots is 6 and the product of these two roots is 25. Therefore a quadratic equation with leading coefficient of 1, the sum of whose roots is 6 and the product of whose roots is 25 is $x^2 - 6x + 25 = 0$. The required cubic equation is $(x-2)\left(x^2 - 6x + 25\right) = x^3 - 8x^2 + 37x - 50 = 0$. So $b = -8$, $c = 37$, $d = -50$ and $c - b - d = 95$.

Alternate solution: In the equation $x^3 + bx^2 + cx + d = 0$, $b = -$ sum of the roots, $c =$ the sum of the products of the roots taken two at a time, and $d = -$ product of the roots. Therefore $b = -(3 + 4i) + (3 - 4i) + 2 = -8$, $c = 2(3 + 4i) + 2(3 - 4i) + (3 + 4i)(3 - 4i) = 6 + 8i + 6 - 8i + 25 = 37$, and $d = 2(3 + 4i)(3 - 4i) = 6$.

14. **197** Apply the base change rule for logs:

$$log_{243} x + log_{81} x + log_{27} x + log_9 x + log_3 x = \frac{a}{b} log_3 x = \frac{log\, x}{log\, 243} + \frac{log\, x}{log\, 81} + \frac{log\, x}{log\, 27} + \frac{log\, x}{log\, 9} + \frac{log\, x}{log\, 3} =$$

$$\frac{log\, x}{log\, 3^5} + \frac{log\, x}{log\, 3^4} + \frac{log\, x}{log\, 3^3} + \frac{log\, x}{log\, 3^2} + \frac{log\, x}{log\, 3} = \frac{log\, x}{5 log\, 3} + \frac{log\, x}{4 log\, 3} + \frac{log\, x}{3 log\, 3} + \frac{log\, x}{2 log\, 3} + \frac{log\, x}{log\, 3} =$$

$\dfrac{\log x}{\log 3}\left(\dfrac{1}{5} + \dfrac{1}{4} + \dfrac{1}{3} + \dfrac{1}{2} + 1\right) = \dfrac{137}{60}\log_3 x$. So $a = 137$ and $b = 60$ and $a + b = 197$.

15. **1** Since the line $y = 2x$ is tangent to the parabola $y = ax^2 + 1$, the system of equations can have only one solution. Set $2x = ax^2 + 1 \rightarrow ax^2 - 2x + 1 = 0$. This equation has only one solution so the discriminant must equal 0. The discriminant, $b^2 - 4ac = (-2)^2 - 4(a)(1) = 4 - 4a = 0 \rightarrow a = 1$.

Grade Level 12 - NMT 2014 **Solutions**

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Mathletics - NMT 2014 **Solutions**

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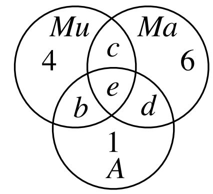
Team Problem Solving - NMT 2014 **Solutions**

1. **3** Since each category of 2 students and three students must be different and no two contain the same number of students, the subgroups b, c, d, and e contain 2, 3, 5, and 7. You could use trial and success to test which number must be in section e or set up 3 equations and solve. We know that $1 + 4 + 6 + b + c + d + e = 28$ or $b + c + d + e = 17$. We also know that $1 + b + d + e = 4 + b + c + e = 6 + c + d + e$. Using these equations we find that $c = b - 5$, $d = b - 2$, and $e = 24 - 3b$. We can deduce that b is the largest so $b = 7$, $c = 2$, $d = 5$, and $e = 3$.



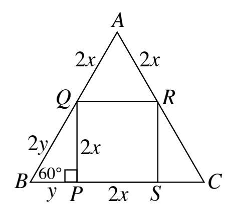
2. **9** Solve for b: $5\cdot(4b + 3) = 2(b^2) + 3b + 6 \rightarrow 20b + 15 = 2b^2 + 3b + 6 \rightarrow 2b^2 - 17b - 9 = 0 \rightarrow (2b + 1)(b - 9) = 0$. Since $b > 0$ and an integer, $b = 9$.

3. **17** Since OP and OQ are lengths of the radius, they are equal. Solve $\sqrt{x^2-111}=\sqrt{110+4x}$. Square both sides and set to zero: $x^2-4x-221=0 \rightarrow (x - 17)(x + 13) = 0$ so $x = 17$.

4. **0** The number $11 = (-4) + (-3) + (-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5 + 6$. Their product is 0.

5. **369** $\left|\left||-123|-(-123)\right|-(-123)\right|=\left||123+123|+123\right|=369$.

6. **576** Arrange the square so that one side is on a side of the triangle and the other vertices are on the other 2 sides of the triangle as in the diagram. Let a side of the square be $2x$ and realize that ΔPQB is a 30°-60°-90° triangle. If we let $PB = y$, then $2x = y\sqrt{3} \rightarrow y=\frac{2x}{\sqrt{3}}=\frac{2x\sqrt{3}}{3}$. We also know that $2x + 2y = 8\left(2\sqrt{3}+3\right)$.



Substitute to eliminate y: $2x+\frac{4x\sqrt{3}}{3}=8\left(2\sqrt{3}+3\right) \rightarrow x\left(\frac{6+4\sqrt{3}}{3}\right)=8\left(2\sqrt{3}+3\right) \rightarrow$

$x=8\left(2\sqrt{3}+3\right)\cdot\frac{3}{6+4\sqrt{3}}=8\left(2\sqrt{3}+3\right)\cdot\frac{3}{2\left(2\sqrt{3}+3\right)}=12$. Since $x = 12$, $2x = 24$ and the area of the square is $24^2 = 576$.

7. **19** The prime factorization of $2014 = 2\cdot19\cdot53$. Since $53^2 = 2809$ the sum of $2 + 8 + 0 + 9$ is 19.

8. **3** The only values that satisfy the equation are $x = 9$ and $x = -6$ and their sum is 3.

9. **278** Convert 12 minutes and 6 seconds into seconds: $12 \times 60 + 6 = 726$. For Jose to write the numbers 1 - 9 he used 9 seconds. Numbers 10 - 99 required $2 \times 90 = 180$ additional seconds. Numbers 100 - 199 used $3 \times 100 = 300$ seconds more for a total thus far of $9 + 180 + 300 = 489$. We have 726 - 489 = 237 digits (seconds) to go. Every number in the 200's is 3 digits long and 237/3 = 79. Therefore Jose wrote from 200 - 278 to get the 79 additional numbers so $n = 278$.

10. **90** The graph is a square with vertices at $\left(\pm3\sqrt{5},0\right)$ and $\left(0,\pm3\sqrt{5}\right)$. Therefore the diagonal, d, of the square is $6\sqrt{5}$ and the area of a square is $A=d^2/2=\left(6\sqrt{5}\right)^2/2=90$.

11. **73** Let p, q, and r be $3x$, $4x$, and $5x$ respectively. $\frac{2(3x)+3(4x)}{5(4x)+7(5x)}=\frac{18x}{55x}=\frac{18}{55}=\frac{a}{b}$. Since the gcd of 18 and 55 is 1, $a = 18$, $b = 55$ and $18 + 55 = 73$.

12. **7** The length of the hypotenuse $AB = 50$. Point M is 25 units from B. ΔAQB is similar to ΔCAB so it must have sides proportional to 3:4:5. Since the hypotenuse of ΔAQB is 30, the two legs must be 18 and 24 with $BQ = 18$. Therefore the distance from M to Q is $25 - 18 = 7$.

13. **259** Apply the fact that $(a+b)^3 = a^3+3a^2b+3ab^2+b^3 = a^3+b^3+3ab(a+b)$ to substitute the values 7 and 4. Therefore $7^3 = a^3 + b^3 + 3(4)(7) \rightarrow 343 = a^3 + b^3 + 84$ so $a^3 + b^3 = 259$. An alternate approach is to use $a^3+b^3=(a+b)(a^2-ab+b^2)$ and $(a+b)^2=a^2+2ab+b^2$.

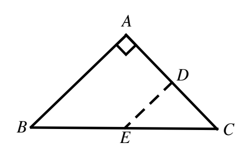
14. **35** From each vertex one can draw 7 diagonals in a decagon. Since there are 10 vertices, one might think that the answer is $7 \times 10 = 70$. This counts each diagonal twice, so divide by 2 to get 35. *Alternate approach*: The number of segments connecting two vertices of a decagon is ${}_{10}C_2 = 45$. Subtract the 10 edges to find that 35 are diagonals.

15. **760** Let a, b, and c be the three integers. Then $a + b + c = 2014$, $a/b = 3/5$, and $a/c = 4/7$. Solve the last two equations for b and c respectively to get $b = 5a/3$ and $c = 7a/4$. Substitute these values into the sum equation: $a + 5a/3 + 7a/4 = 2014 \rightarrow 12a + 20a + 21a = 2014 \cdot 12 \rightarrow a = 2014 \cdot 12/53 = 456$. Since the problem asks for the second number, b, multiply this by 5/3 to get 760.

16. **140** Notice that $\left(x-\frac{1}{x}\right)^2 = x^2-2+\frac{1}{x^2}$ and that $\left(x+\frac{1}{x}\right)^2 = x^2+2+\frac{1}{x^2}$. This means that the square of the sum of a number and its reciprocal is always 4 more than the square of the difference between the same number and its reciprocal. So $12^2 - 4 = 140$ is our answer.

17. **15** Realize that $y = 1$ implies that $x = 100$ so $(100, 0)$ is a lattice point. The slope of the line is $-20/14 = -10/7$. This means that for every 10 added to y, 7 must subtracted from x. Since x and y must be positive and there are fourteen 7's in 100, an additional 14 integer values can be subtracted from x before becoming negative. Thus there are 15 integer pairs or lattice points that satisfy the equation.

18. **34** Since the perpendicular bisector of one side is also the bisector of the base of the isosceles triangle, it must be parallel to the third side, so $\overline{AB} \perp \overline{AC}$, so ΔABC is a right isosceles triangle. Since a leg is $17\sqrt{2}$, the hypotenuse is $17\sqrt{2}\cdot\sqrt{2} = 34$.



19. **136** $\Delta AEP \cong \Delta CBP$ by AAS. Therefore $PE = PB$. Let $PB = x = PE$ so $AP = 32 - x$. Apply the Pythagorean theorem in ΔAEP to set up the equation $8^2 + x^2 = (32-x)^2$. Square and simplify to get $x = 15$ and $AP = 32 - 15 = 17$. Quadrilateral $APCQ$ is a parallelogram (and a rhombus) so its area is $A_{\square} = bh = 17 \cdot 8 = 136$.

20. **185** Since $f(x) = x^2 + 9x + w$ and $f(w) = -25$, we can substitute to find $f(w) = w^2 + 9w + w = -25$. It follows that $w^2 + 10w + 25 = 0$ so $(w + 5)^2 = 0$ and $w = -5$. Thus $f(10) = 10^2 + 9(10) - 5 = 185$.

Tie Breakers - NMT 2014 **Solutions**

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