**Grade Level 9 - NMT 2013** **Solutions**

1. **7** The graph of the given equation is a parabola and its maximum value lies on the axis of symmetry. Therefore, since $x = \frac{-b}{2a} = \frac{-4}{2(-2)} = 1$, the maximum for $y = -2(1)^2 + 4(1) + 5 =$ **7**.

2. **13** Since $\frac{1}{4}\% = 0.0025$ we can multiply that by 5200 to get **13**. Alternatively move the percent sign to the 5200 to get 52 and then take ¼ of 52 to get 13.

3. **20** After 1 hour, the faster car is 15 miles ahead. After another 1/3 of an hour, it would be an additional 5 miles ahead and 15 + 5 = **20**.

4. **22** Let $n = 0.0\overline{45}$ so $100n = 4.5\overline{45}$. Subtract the first equation from the second equation to get $99n = 4.5 \rightarrow n = 4.5/99 = 45/990 = 1/22$. The reciprocal of our result is **22**.

5. **360** The prime factors of each number are: $12 = 2^2 \cdot 3$, $45 = 3^2 \cdot 5$, and $72 = 2^3 \cdot 3^2$. Therefore, the least common multiple is $2^3 \cdot 3^2 \cdot 5 =$ **360**.

6. **22** Multiply the second equation by 22 to eliminate the fractions. The result is $2x - 11y = 0$. Subtract this equation from the first equation to eliminate the $x$ variable and to get $12y = 24 \rightarrow y = 2$. Back substitute to find $x = 11$ and $2 \cdot 11 =$ **22**.

7. **20** Let $x$ = the manufacturer's price so $x + 0.1x = 1.1x$ = distributer's price. The retail price is $1.1x + .30(1.1x) = 1.3(1.1x) = 28.60 \rightarrow x =$ **20**.

8. **7** The units-digit of the powers of 47 form a repeating pattern of 4-digits; 7, 9, 3, and 1. Therefore the $47^{2013}$ will produce the same units-digit as $47^1$ which is **7**.

9. **90** The area of the shaded region is the area of the outer square minus the sum of the area of the inner square and the areas of the four semi-circles. Since the radius of each semi-circle is 3, the length of a side of square $ABCD = 3 + 6 + 3 = 12$ and its area is 144. The areas of the 4 semi-circles is equivalent to the area of 2 circles $= 2 \cdot \pi(3)^2 = 18\pi$. Since the area of the inner square is $6 \cdot 6 = 36$, the shaded area is $144 - (36 + 18\pi) = 108 - 18\pi$ and $108 + (-18) =$ **90**.

10. **4** There are 4 choices for the thousands-digit, 5 choices for the hundreds-digit, and 8 remaining choices for the two other digits. Using the basic counting principle, $4 \times 5 \times 8 \times 7 = 1120$. The sum of the digits in this number is **4**.

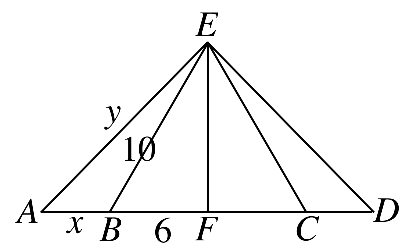
11. **12** Number the cards G1, G2, G3, G4, G5, Y3, Y4, Y5, and Y6. Since colors alternate and G5 must divide into a Y evenly, G5 and Y5 must be next to each other and at one end of the stack. Since G4 must divide a Y number, it must be next to Y4 and at the other end of the stack. Next to Y5 must be G1, the only remaining G number to divide 5. At the other end, next to Y4, must be G2. This leaves the three middle cards as Y3, G3, Y6 and the sum of these numbers is **12**. The order of the cards can be G5, Y5, G1, Y3, G3, Y6, G2, Y4, and G4.

12. **17** Expand the numbers so NMT12 + NMT13 = 10000N + 1000M + 100T + 12 + 10000N + 1000M + 100T + 13 = 20000N + 2000M + 200T + 25 = 129425 → 200N + 20M + 2T = 1294 → 100N + 10M + T = 647 so N = 6, M = 4, and T = 7 and N + M + T = 6 + 4 + 7 = **17**.

13. **900** Row one uses 3 toothpicks; row two uses 6 toothpicks; and row three uses 9 toothpicks. This pattern continues so in row $n$ uses $3n$ toothpicks. Therefore in the 150$^{\text{th}}$ row, there will be 3(24) = 72 toothpicks. Now we must sum the numbers 3 + 6 + 9 + … + 66 + 69 + 72. Notice that 3 + 72 = 6 + 69 = 9 + 66 = … = 75. Since there are 12 pairs of numbers with this sum the total for all the numbers is $75 \times 12 =$ **900**.

14. **58** Method 1: Let $a + b = 8$ and $ab = 3$. We wish to calculate $a^2 + b^2$. Notice that $(a+b)^2 = a^2 + 2ab + b^2 \rightarrow a^2 + b^2 = (a+b)^2 - 2ab = (8)^2 - 2(3) = \mathbf{58}$.
Method 2: Since $b = 8 - a$, $ab = a(8 - a) = 3 \rightarrow a^2 - 8a + 3 = 0$. If you now apply the quadratic formula you will find that $a = 4 \pm \sqrt{13}$ and back substitute to get that $b = 4 \mp \sqrt{13}$ respectively. Finally, $a^2 = \left(4 \pm \sqrt{13}\right)^2 = 29 \pm \sqrt{13}$ and $b^2 = 29 \mp \sqrt{13}$ so their sum is 58.

15. **9** Let the foot of the altitude from $E$ to $\overline{AB}$ be point $F$. Then $F$ will also be the midpoint of $\overline{AB}$. Since $BF = 6$ and $BE = 10$, by the Pythagorean theorem, $EF = 8$. Let $AB = CD = x$ and $AE = DE = y$. The perimeter of $\Delta AED$ = 2(perimeter of $\Delta AEF$) so $2x + 2y + 12 = 2(32) \rightarrow x + y = 26 \rightarrow y = 26 - x$. Apply the Pythagorean theorem on $\Delta AEF$ so $(x+6)^2 + 8^2 = y^2 \rightarrow (x+6)^2 + 8^2 = (26-x)^2$.




Simplifying this equation yields $12x + 100 = 676 - 52x \rightarrow 64x = 576 \rightarrow x = 9$.

**Grade Level 10 - NMT 2013** **Solutions**

1. **21** The $x$-coordinate can be found using $x = \dfrac{-b}{2a} = \dfrac{-4}{2\left(\frac{2}{3}\right)} = -3$. The $y$-coordinate is found by substituting this result into the equation to get $y = \frac{2}{3}(-3)^2 + 4(-3) - 1 = -7$ and (–3)(–7) = **21**.

2. **360** Use the prime factorization of each number: 8 = 2·2·2, 18 = 2·3·3, and 20 = 2·2·5. The LCM is equal to 2·2·2·3·3·5 = **360**.

3. **8** The prime factorization of 2013 is 3·11·61, so the list of factors consists of 1, 3, 11, 61, 2013, 3·11, 3·61, and 11·61. Therefore there are a total of **8** factors.

4. **16** Square both sides of the equation and then isolate $x$: $\sqrt{x} - \frac{11}{9} = \frac{25}{9} \rightarrow \sqrt{x} = \frac{36}{9} = 4 \rightarrow x = \mathbf{16}$.

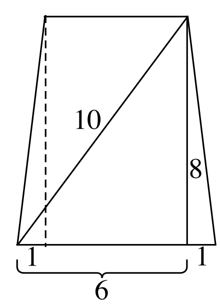
5. **10** The slope of the line is the negative reciprocal of 1/3, so $m = -3$. Using $y = mx + b$ and the given point, the $y$-intercept can be found: $-2 = -3(4) + b \rightarrow b = \mathbf{10}$.

6. **480** The greatest perfect cube that is a factor of $640x^{20}$ is $64x^{18}$. Therefore $\sqrt[3]{640x^{20}} = \sqrt[3]{64x^{18}} \cdot \sqrt{10x^2} =$ $4x^6 \cdot \sqrt[3]{10x^2}$ .

7. **229** There are 299 – 18 + 1 = 282 numbers in the data set. The first 141 numbers make up the lower half so the upper half starts at the 142nd number that includes 159 through 299. The midpoint of the upper half is the upper quartile. Since they are consecutive integers, this midpoint will be the average of 159 and 299 = (159 + 299)/2 = **229**.

8. **780** 125° is the only obtuse angle that is a perfect cube. The required sum is 5 + 25 + 125 + 625 = **780**.

9. **48** The Pythagorean theorem can be used with the diagonal as the hypotenuse and an altitude as a leg. The missing side is 6. This means that the bases of the two small right triangles on each side of the central rectangle are each 1 so that the longer base is 7. The area of a trapezoid can be found as follows:
$A = \frac{h}{2}(b_1 + b_2) = \frac{8}{2}(5+7) = \mathbf{48}$ .




10. **800** Multiply out each side independently and combine like terms to get: $10x^2 - 24x = 2x^2 + 3x - 9 \rightarrow$ $8x^2 - 27x + 9 = 0 \rightarrow (8x-3)(x-3) = 0$. So $x = 3/8$ or $x = 3$. To answer the question, convert $\dfrac{3}{3/8}$ into a percent by multiplying by 100 to get **800**.

11. **698** There are 7 ½ eight-minute segments in 1 hour. Multiply 93 million by 7.5 to obtain 697.5 million which rounds to 698 million. Multiply by $10^{-6}$ to obtain **698**.

12. **24** Use the Pythagorean theorem

13.

14.

15.

**Grade Level 11 - NMT 2013** **Solutions**

1. **39** $5\log_5 125+4\log_4\frac{1}{64}+6\log_a a^6=5\cdot 3+4\cdot(-3)+6\cdot 6=\mathbf{39}$ .

2. **29** The point (20, 21) can be plotted on the Argand plane. The absolute value or modulus of a complex number is the distance from the origin of the plane to the plotted point. $\sqrt{20^2+21^2}=\sqrt{841}$ = **29**.

3. **60** The measure of an angle formed by two tangents to a circle from a common exterior point is one-half the absolute value of the difference of the measures of the intercepted arcs. Since the measure of the major arc is 240°, the measure of the minor arc is 120°. The measure of the required angle is $\frac{1}{2}(240-120)$ = **60**.

4. **240** Use $k=\frac{1}{2}ab\sin c=\frac{1}{2}\cdot 24\cdot 40\cdot\sin 30°=\frac{1}{2}\cdot 24\cdot 40\cdot\frac{1}{2}=\mathbf{240}$ . Alternatively, drop an altitude from vertex $A$ to $\overline{BC}$ . Using ratios from 30°-60°90° triangles, the length of the altitude is 12 and $A=\frac{1}{2}bh=\frac{1}{2}\cdot 40\cdot 12=\mathbf{240}$ .

5. **34** Since $x^2-3x-28=(x-7)(x+4)\ge 0$ so $x \ge 7$ or $x \le 4$. According to the conditions in the problem, $x$ = 7, 8, 9, or 10. The required sum is **34**.

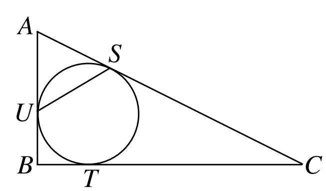
6. **30** Suppose $x$ people board the elevator on the first floor. After the elevator stops on the second floor and five people exit, $x$ – 5 remain. After the elevator stops on the third floor, $\frac{3}{5}(x-5)$ exit and $\frac{2}{5}(x-5)$ remain. Finally, $\frac{1}{5}\cdot\left(\frac{2}{5}(x-5)\right)=2$ or $x$ = **30**.

7. **280** Of the first 600 positive integers, 600/3 = 200 are divisible by 3, 600/5 = 120 are divisible by 5, and 600/15 = 40 are divisible by both 3 and 5. Therefore 200 + 120 – 40 = **280** are divisible by 3 or 5.

8. **2** If graphs of inverse functions intersect, then they intersect on the line whose equation is $y = x$. Solve $f(x)=x^2-3x+4=x \to x^2-4x+4=0 \to x=\mathbf{2}$.

9. **2** Square both sides to get $k=3-\sqrt{5}-2\sqrt{(3-\sqrt{5})(3+\sqrt{5})}+3+\sqrt{5}=6-2\sqrt{4}=\mathbf{2}$ .

10. **38** A 5-12-13 triangle is a right triangle. Without loss of generality, let $AC$ = 5, $BC$ = 12, and $AB$ = 13. Let $S$, $T$, and $U$ be points of tangency on $\overline{AC}$, $\overline{BC}$, and $\overline{AB}$ respectfully. Now $AS = AU$, $BU = BT$, and $CS = CT$ because tangent segments to a circle from the same external point are congruent. Note that $AS = AU$ = 3, $BU = BT$ = 2, and $CS = CT$ = 10. Draw $\overline{SU}$ and use the Law of Cosines in $\Delta ASU$: $(SU)^2=3^2+3^2-2\cdot 3\cdot 3\cos A$ = 9 + 9 – 18(5/13) = 144/13 and $SU=\frac{12\sqrt{13}}{13}$ .




Thus 12 + 13 + 13 = **38**. Alternatively, we can call the center of the circle $O$ and draw radii $\overline{OS}$ and

$\overline{OU}$. Note that these radii (each having length 2) are perpendicular to $\overline{AC}$ and $\overline{AB}$ respectively. Thus, in quadrilateral *AUOS*, $\angle ASO$ and $\angle AUO$ are right angles making that quadrilateral inscribable in a circle. Ptolemy's theorem says that the product of the lengths of the diagonals of a quadrilateral inscribed in a circle is the sum of the products of the lengths of the two pairs of opposite sides. So, $SU{\cdot}AO = AU{\cdot}SO + AS{\cdot}UO$. Thus $SU\cdot\sqrt{13}=3\cdot 2+3\cdot 2$ giving the same result.

11. **196** The number of five-person committees that can be chosen from ten people is $\binom{10}{5}=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2\cdot 1}=252$. The number of five-person committees that can be chosen from ten people that include Heckle and Jeckle is $\binom{8}{3}=\frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1}=56$. The required committee can be chosen in 25 – 56 = **196** ways. Alternatively, we can consider three cases. If neither Heckle nor Jeckle is on the committee, there are $\binom{8}{5}=56$ five-person committees. If Jeckle is on the committee and Heckle is not, there are $\binom{8}{4}=70$ five-person committees. The same result occurs when Heckle is on and Jeckle is not on the committee.

12. **193** The area of a rhombus is one-half the product of its diagonals. So the area of our rhombus is $\frac{1}{2}\cdot 6\cdot 8=24$. Alternatively, call our rhombus *ABCD*, with the longer diagonal $\overline{AC}$, and shorter diagonal $\overline{BD}$ that intersect at right angles at point *E*. *AE* = 4, *BE* = 3, and *AB* = 5. So the area of each right triangle is 6 and the area of the rhombus is 24. An altitude from point *E* to $\overline{AB}$ is also a radius of circle *E*. Use the area of $\Delta AEB = \frac{1}{2}\cdot 5\cdot r=\frac{1}{2}\cdot 3\cdot 4$ to get $r = 12/5$. The area of the requested region is $24 - 144\pi/25$ and the required sum is 24 + 144 + 25 = **193**.

13. **168** Since $\cot x+\cot y=\frac{1}{\tan x}+\frac{1}{\tan y}=\frac{\tan x+\tan y}{\tan x\tan y}=\frac{24}{\tan x\tan y}=28$**,** $\tan x\tan y=\frac{6}{7}$. Therefore $\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{24}{1-6/7}=\mathbf{168}$.

14. **12** $(x-3)^4-(x-3)=(x-3)\left((x-3)^3-1\right)=(x-3)\left((x-3)-1\right)\left((x-3)^2+(x-3)+1\right)=$ $(x-3)(x-4)\left(x^2-5x+7\right)=0$. The requires sum is 5 + 7 = **12**.

15. **8** The length of the side of the first square is 2 and its area is 4. The length of the side of the second square is $\sqrt{2}$ and its area is 2. The length of the side of the third square is 1 and its area is 1. The length of the side of the fourth square is $1/\sqrt{2}$ and its area is 1/2. 4 + 2 + 1 + 1/2 + … is a geometric series with ratio of 1/2 that converges to $\frac{a}{1-r}=\frac{4}{1-1/2}=\mathbf{8}$.

**Grade Level 12 - NMT 2013** **Solutions**

1. **697** Method 1: $a_1 = 39$ , $a_1 + 30d = 2013$. Solve for $d = \dfrac{1974}{30}$. $a_{11} = 39 + \dfrac{1974}{30}(10) = 39 + \dfrac{1974}{3} = \mathbf{697}$.
Method 2: Since $a_1$, $a_{11}$, $a_{21}$, and $a_{31}$ are evenly spaced terms in the arithmetic sequence, they form an arithmetic subsequence. The spacing between terms of this subsequence is $\dfrac{2013-39}{3} = 658$ and so $a_{11} = 39 + 658 = \mathbf{697}$.

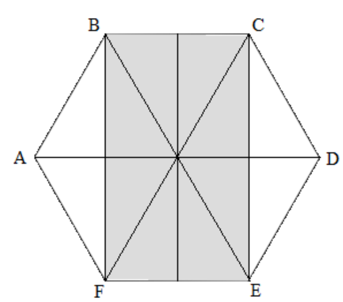
2. **662** Adding the first and third equations yields $4x + 6y + 8z = k + 2013$. Factoring gives $2(2x+3y+4z) = 2013 + k$ and so $2(1337) = 2013 + k$. Solve for $k = \mathbf{661}$.

3. **867** Using the chain and product rules will work, however, if we multiply out $f(x)$ before differentiating, we get $f(x) = x^3 - 2x\sqrt{x} + 1 + 2x^{\frac{3}{2}} = x^3 + 1$. From this, we get $f'(x) = 3x^2$ and so $f'(17) = 3(289) = \mathbf{867}$.

4. **2** Let $k = \sqrt{3}$.
Method 1: $\displaystyle\sum_{n=672}^{2013} \log_k\left(\frac{n}{n-1}\right) = \log_k\left(\frac{672}{671}\right) + \log_k\left(\frac{673}{672}\right) + \log_k\left(\frac{674}{673}\right) + ... + \log_k\left(\frac{2013}{2012}\right) =$
$\log_k\left(\dfrac{672}{671}\cdot\dfrac{673}{672}\cdot\dfrac{674}{673}\cdot...\cdot\dfrac{2013}{2012}\right) = \log_k\left(\dfrac{2013}{671}\right) = \log_k 3$. Finally, if $\log_{\sqrt{3}} 3 = w$ then
$\left(\sqrt{3}\right)^w = 3 \rightarrow 3^{w/2} = 3^1$ so $w = \mathbf{2}$.
Method 2: $\log_k\left(\dfrac{n}{n-1}\right) = \log_k n - \log_k(n-1)$, so $\displaystyle\sum_{n=504}^{2012} \log_k\left(\frac{n}{n-1}\right) =$
$(\log_k 672 - \log_k 671) + (\log_k 673 - \log_k 672) + ... + (\log_k 2013 - \log_k 2012)$. This series telescopes, leaving only $\log_k 2013 - \log_k 671 = \log_k\left(\dfrac{2013}{671}\right)$. Continue as in method 1.

5. **60** Factor (the numerator by difference of squares, the denominator by grouping):
$$\lim_{x\to5}\lim_{y\to3}\frac{(x-5)(x+5)(y-3)(y+3)}{(x-5)(y-3)} = \lim_{x\to5}\lim_{y\to3}(x+5)(y+3) = (5+5)(3+3) = \mathbf{60}.$$

6. **666** Method 1: Let $s$ be the length of a side of the hexagon. Multiplying the formula $A = s^2\dfrac{\sqrt{3}}{4}$ for the area of an equilateral triangle by 6 gives us the equation $s^2\dfrac{3\sqrt{3}}{2} = 999 \rightarrow s^2 = \dfrac{666}{\sqrt{3}}$. Since triangle BCF is 30-60-90, $BF = s\sqrt{3}$ and so the area of BCEF is
$s^2\sqrt{3} = \dfrac{666}{\sqrt{3}}\sqrt{3} = \mathbf{666}$.



Method 2: Divide the hexagon into 12 congruent triangles, as shown. Since the rectangle is made up of 8 out of the 12 triangles, the rectangle has an area which is two-thirds of the area of the hexagon, or **666**.

7. **17** Since the slope of the given line is 37 and the tangent line does not intersect it, the slope of the tangent line must also be 37. Because $\frac{dy}{dx}=2x+3=37$, the tangent line must have been drawn where $x=\mathbf{17}$.

8. **503** Since $2012x=2\cdot1006x$, we can use the double-angle formula for sine:
$f(x)=\frac{1}{2}(2\sin1006x\cos1006x)\frac{1}{\cos1006x}=\sin1006x \rightarrow f'(x)=1006\cos(1006x) \rightarrow$
$f'\left(\frac{\pi}{3018}\right)=1006\cos\left(1006\frac{\pi}{3018}\right)=1006\cos\left(\frac{\pi}{3}\right)=1006\left(\frac{1}{2}\right)=\mathbf{503}$.

9. **54** Let $y=x^2-7x$, giving us the equation $y^2+22y+120=0$, which can be factored to yield $y=-10$ or $y=-12$. $x^2-7x=-10 \rightarrow x^2-7x+10=0 \rightarrow (x-2)(x-5)=0 \rightarrow x=2, x=5$ and $x^2-7x=-12 \rightarrow x^2-7x+12=0 \rightarrow (x-3)(x-4)=0 \rightarrow x=3, x=4$. The roots are 2, 3, 4, and 5, and so $2^2+3^2+4^2+5^2=\mathbf{54}$.

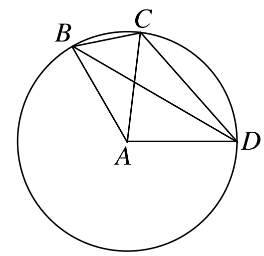
10. **171** Since both *f* and *g* are linear, $f\circ g$ is composed with itself, the result is the identity function, which means that $f\circ g$ is its own inverse. A linear function is only its own inverse if $y=(f\circ g)(x)$ is symmetric with respect to the line $y=x$, meaning that the slope of $y=(f\circ g)(x)$ has to be $\pm1$. Let $g(x)=mx+b \rightarrow (f\circ g)(x)=\frac{17-2(mx+b)}{342}=\frac{17-2mx-2b}{342}=\frac{m}{171}x+\frac{17-2b}{342}$. Therefore, $\frac{m}{171}=\pm1 \rightarrow m=\pm171 \rightarrow |m|=|g'(x)|=\mathbf{171}$.

11. **0** <u>Method 1</u>: $f'(x)=\frac{(x+1)\frac{1}{2\sqrt{x}}-\sqrt{x}}{(x+1)^2}=0$ when $\frac{x+1}{2\sqrt{x}}-\sqrt{x}=0 \rightarrow \frac{x+1}{2\sqrt{x}}=\sqrt{x} \rightarrow x=1$ and $f(1)=\frac{1}{2}$.
However, use of the first or second derivative tests shows that $x = 1$ is a relative *maximum* of *f*. Therefore, the absolute minimum must occur at an endpoint, and, checking that $f(0) = 0$ and $\lim_{x\to\infty} f(x)=0$, the absolute minimum value of *f* is **0**.
<u>Method 2</u>: Note that the domain of *f* is the nonnegative real numbers and, over this domain, $f(x)>0$. Since $f(0) = 0$, this value must be the absolute minimum of the function.

12. **816** <u>Method 1</u>: $\frac{16\cdot15\cdot14}{3\cdot2\cdot1}+\frac{16\cdot15}{2\cdot1}+\frac{17\cdot16}{2\cdot1}=560+120+136=\mathbf{816}$.
<u>Method 2</u>: If we consider the positions of ${}_{16}C_{13}$, ${}_{16}C_{14}$, and ${}_{17}C_{15}$ in Pascal's triangle, we note, based on the rule used to construct the triangle, that ${}_{16}C_{13} + {}_{16}C_{14} + {}_{17}C_{15} = {}_{18}C_{15} = 816$.

13. **516** Note that no matter where the first two points are located, the third and fourth points form a square along with the two original points. After four points, the cycle begins to repeat, and so $P_1 = P_5 = P_9 = \ldots = P_{2013}$. Therefore, the answer is 208 + 308 = **516**.

14. **194** Method 1: Determine the equations of the lines and find their point of intersection, which is (0, **194**).
Method 2: Since $y$ is an even function of $x$ and the tangent lines are drawn to points that are symmetric with respect to the $y$-axis, the lines must intersect on the $y$-axis. Therefore, it suffices to find the $y$-intercept of either tangent line. Using $x = 3$: $y' = -4x^3 + 10x\big|_{x=3} = -78$, so the tangent line equation is $y + 40 = -78(x - 3)$, which has a $y$-intercept of 194.




15. **28** Since $AB = AC = AD$, $A$ is the circumcenter of $\Delta BCD$. Since $\angle BAD$ is a central angle, $m\,\widehat{BCD} = 120°$, the measure of major $\widehat{BD}$ =240°, and $m\angle BCD = 120°$ (inscribed angle). Let $x = BD$. By the law of cosines,
$x^2 = 12^2 + 20^2 - 2(12)(20)\cos 120° \to x^2 = 784 \to x = \mathbf{28}$.

**Mathletics - NMT 2013** **Solutions**

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

**Team Problem Solving - NMT 2013 Solutions**

1. **66** Start with 3-down or 1-across since there is only one possible number for each, 997 and 1597 respectively. Now 2-down has only one possibility, 512. Unfortunately 1-down has two possibilities, either 121 or 169. If you try 121 then 5-across is 219_, which cannot be divisible by 11 for any units-digit. Therefore 1-down must be 169 and 5-across is 6193. The last space must be a 7 since 4-down is palindromic. The horizontal numbers are 1597, 6193, and 9277 and the sum of the digits is **66**.

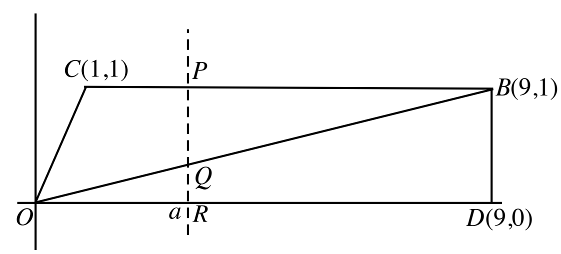
2. **271** There are $9 \cdot 9 \cdot 9 - 1 = 728$ integers that do not contain a 7 (the –1 removes the integer 000). Subtract this from the 999 positive integers less than 1000, so 999 – 728 = **271** integers contain a 7. Alternatively one can count the number of integers that contain a 7 directly. There are 19 such numbers from 1 – 99 as well as in each group of 100 integers except for the 700 – 799 group, all of which contain a 7. Thus we have 9(19) + 100 = **271**.

3. **8** Suppose, without loss of generality, that the shorter side of the rectangle has a length of 1. Since the diagonal of the rectangle is the diameter of the circle, the radius of the circle is $\frac{\sqrt{101}}{2}$. Thus

$$\frac{\text{area of circle}}{\text{area of rectangle}} = \frac{\pi\left(\sqrt{101}/2\right)^2}{10} \approx 7.9 = \mathbf{8}$$ to the nearest whole number.

4. **720** Let the edges of the solid be $l$, $w$, and $h$. Therefore, $l^2 + w^2 = 10^2, w^2 + h^2 = 17^2$ and $l^2 + h^2 = \left(\sqrt{261}\right)^2$. Subtract equation 1 from equation 2 to get $h^2 - l^2 = 189$. Add this result to the third equation to find that $h = 15$. Substitute into each of the original equations to find that $w = 8$ and $l = 6$. Thus the volume = $lwh$ = (6)(8)(15) = **720**. Alternatively, one can add all three original equations and then divide by 2 to get $l^2 + w^2 + h^2 = 325$. Now subtract each of the given equations to get the square of each variable and then square root each result to find $l$, $w$, and $h$.

5. **3** The area of the ΔCOB is (1/2)(1)(8) = 4. Thus we want the area of Δ$BPQ$ = 2. Drop a perpendicular segment from $B$ to the $x$-axis. Δ$QRO$ ~ Δ$BDO$ so $QR/RO = BD/DO = 1/9$. Since $RO = a$, $QR = a/9$, $PQ = 1 - a/9$, and $PB = 9 - a$. Solve for $a$ in the area equation $A = \frac{1}{2}\left(1 - \frac{a}{9}\right)(9 - a) = 2$. Multiply and



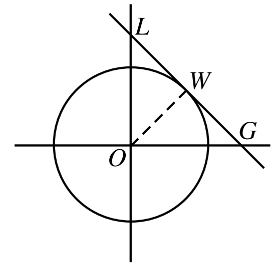

factor to get $a^2 - 18a + 45 = 0 \rightarrow (a-3)(a-15) = 0$ so $a$ = **3**. Of course one can just substitute some reasonable integers and test them in the initial area equation. An alternative approach is to apply coordinate geometry to write the equation of $\overleftrightarrow{OB}$ and then find the coordinates of $P$ and $Q$. Then continue as above.

6. **497** Both the numerator and denominator are arithmetic sequences. The sum of the numbers in an arithmetic sequence can be found by several different techniques. Applying the formula $S = \frac{n}{2}(a_1 + a_n)$ we find that $S_{\text{num}} = n^2$ and $S_{\text{denom}} = n(n+1)$ so $\frac{n^2}{n(n+1)} = \frac{n}{n+1} = \frac{497}{498}$, $n$ = **497**.

7. **38** Since the graphs of lines symmetric about the line $y = x$ are inverses of each other, the equation of the line symmetric to $y = ax + b$ is $y = x/a - b/a$. Thus the line symmetric to $y = x/2 - 18$ has the equation $y = 2x + 36$ and $2 + 36 =$ **38**.

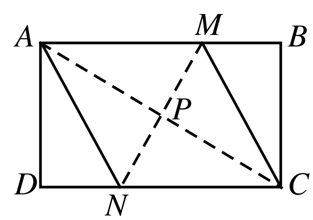
8. **75** Some lesser known operations with proportions include the ideas that if $x/y = a/b$ then $(x + py)/y = (a + pb)/b$ and $x/(y + px) = a/(b + pa)$. Therefore, $x/y = 3/7 \rightarrow (x + 2y)/y = (3 + 14)/7 = 17/7 \rightarrow (x + 2y)/(3x + 7y) = 17/(7 + 3(17)) = 17/58$ and $17 + 58 =$ **75**. Alternatively, one can solve the given proportion for $x$, and then substitute and reduce in the second proportion. This results in $x = 3y/7$ and $(3y/7 + 2y)/(9y/7 + 7y) = (3 + 14)/(9 + 49) = 17/58$ and $17 + 58 = 75$.

9. **9** Let the 4-digit number be $1000s + 100h + 10t + u$ so the number with digits reversed will be $1000u + 100t + 10h + s$. Subtracting these results in $999s + 90h - 90t - 999u$ and the only factor for this result that is independent of the values for $s$, $h$, $t$, and $u$ is **9**.

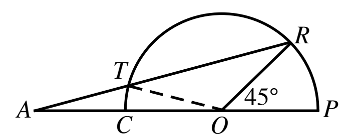



10. **100** There are two possible placements for the line but the resulting triangles are congruent. Let the line be in quadrant I as seen in the diagram. The radius $OW = 10$, $m\angle WOG = 45°$ and $m\angle OWG = 90°$. Therefore $OG = 10\sqrt{2}$ and the area of $\Delta LOG = 0.5(10\sqrt{2})(10\sqrt{2}) =$ **100**.

11. **13** Let $p$ be the price of the item, so $1.04p = n \rightarrow 104p = 100n \rightarrow 8(13)p = 4(25)n \rightarrow 2(13)p = 25n$. Since $p$ does not have to be an integer, the smallest value for $n =$ **13**. When $n = 13$, $p = \$12.50$.




12. **130** The greatest common factor of 120 and 288 is 24. That is $288 = 24(12)$ and $120 = 24(5)$ So $\Delta ABC$ is a 5-12-13 triangle with $AC = 24(13) = 312$ and $AP = 156$. Since the diagonals of a rhombus are perpendicular bisectors of each other, $\Delta APM \sim \Delta ABC$. Use the proportion $MP/AP = BC/AB \rightarrow MP/156 = 120/288$, so $MP = 65$ and $MN = 130$. An alternative approach is to let $MB = x$ so $AM = MC = 288 - x$ and apply the Pythagorean theorem in $\Delta CBM$ to find that $x = 119$. Translate $\overline{MN}$ so that $M$ and $B$ coincide and $N$ moves to $N'$. Apply the Pythagorean theorem in $\Delta BCN'$ noting that $CN' = 169 - 119 = 50$ to find $MN =$ **130**.

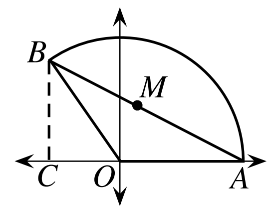



13. **15** Draw $\overline{OT}$, another radius of the circle, so $\Delta ATO$ and $\Delta OTR$ are both isosceles triangles. Let $m\angle CAT = x = m\angle TOC$, then $m\angle OTR = 2x = m\angle ORT$. It follows that $m\angle TOR = 180 - 4x$, so $x + (180 - 4x) = 135 \rightarrow 3x = 45 \rightarrow x =$ **15**. An alternate solution uses the measure of an angle formed by secants intersecting outside a circle is one-half the difference of its two intercepted arcs. Let $m\angle A = x$, so $m\angle TOC$ and $m\overset{\frown}{TC}$ are also $x$ and $m\overset{\frown}{RP} = 45°$ so $x = \dfrac{45 - x}{2} \rightarrow x =$ **15**.

14. **8** Since the circles labeled $P$ and $Q$ are each connected to all the circles but one they must be the numbers 1 and 8 since they are each consecutive to only one other number. Their product is **8**.

15. **92** For the sum of the coins to be at least \$1.25, there must be 5 quarters plus one additional coin. Since order does not matter, $\dfrac{{}_6C_6 + {}_6C_5 \cdot {}_5C_1 + {}_6C_5 \cdot {}_4C_1}{{}_{15}C_6} = \dfrac{1 + 6 \cdot 5 + 6 \cdot 4}{5005} = \dfrac{55}{5005} = \dfrac{1}{91} \rightarrow 91 + 1 =$ **92**.

16. **62** Draw the diagram and add $\overline{BC} \perp \overline{AC}$. Since $m\angle AOB = 125°$, $m\angle BOC =$ 55°. The coordinates of $B$ are $(-100\cos 55°, 100\sin 55°)$. The coordinates of the midpoint of $\overline{AB}$ are $\left(\dfrac{100-100\cos 55°}{2}, \dfrac{100\sin 55°}{2}\right)$ = (21.3211, 40.9576). To the nearest integer, the sum of these coordinates is **62**.




17. **28** Use the base change rule for logs to rewrite the equation as $8\log_x 2 + \frac{1}{2}\log_x 49 - 2\log_x 8 = \log_x b$.

Then apply the other log rules to get $\log_x \dfrac{2^8 \cdot \sqrt{49}}{8^2} = \log_x \dfrac{256 \cdot 7}{64} = \log_x 28$ so $b$ = **28**.

18. **121** Since $\Delta BEC \sim \Delta DEA$ and the ratio of their areas is 16/49 the ratio of corresponding sides $BE/ED$ is 4/7. Since $\Delta BCE$ and $\Delta CED$ share the same altitude the ratio of their areas equals the ratio of the sides to which the common altitude is drawn, so $A_{\Delta BCE}/A_{\Delta DCE} = BE/ED = 4/7$. Therefore, $A_{\Delta CED} = 28$. The areas of $\Delta BAD$ and $\Delta DAC$ are equal since they have the same height and share a base. Subtract the area of $\Delta AED$ from each of those triangles so the areas of $\Delta BEA$ and $\Delta CED$ are equal. The area of the trapezoid is the sum of the four triangle areas or 16 + 49 + 28 + 28 = **121**. [Note: The general formula for the area of the trapezoid based upon the areas of the two given triangles is $A = \left(\sqrt{\text{area 1}} + \sqrt{\text{area 2}}\right)^2$.] An alternative approach involves drawing the altitude, $\overline{FG}$, of the trapezoid through $E$. Since the ratio of the areas of the triangles is 16 to 49, the ratio of corresponding linear measurements is 4 to 7. Let $BC = 4x$, $AD = 7x$, $FE = 4y$, and $EG = 7y$. Let the area of $\Delta BEC = 16 = (4x)(4y)/2$ so $xy = 2$. The area of the trapezoid is $(11y)(4x + 7x)/2$ = **121**.

19. **13** Write out a few more terms to see 5, 9, 4, –5, –9, –4, 5, … . This repeating pattern has a length of six. The sum of the first 96 terms is 0 and the next four terms add to **13**, which is the sum of all one hundred terms.

20. **142** Since $f(x) = ax^7 - bx^5 + cx + 57$, $f(-x) = -ax^7 + bx^5 - cx + 57 = -\left(ax^7 - bx^5 + cx\right) + 57$ and $f(x) - 57 = ax^7 - bx^5 + cx$. Since $f(17) = -28$, $f(17) - 57 = -28 - 57 = -85$ and $f(-17) = -(-85) + 57$ = **142**.

**Tie Breakers - NMT 2013** **Solutions**

1.

2.

3.