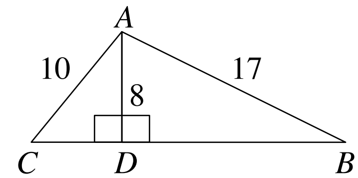
Grade Level 9 - NMT 2012 **Solutions**

1. **30** The value of d is the product of the three constants and 5(–2)(–3) = 30.

2. **26** Simplify: $4\sqrt{98}-\frac{2}{3}\sqrt{72}=4\sqrt{49\cdot 2}-\frac{2}{3}\sqrt{36\cdot 2}=4\cdot 7\sqrt{2}-\frac{2}{3}\cdot 6\sqrt{2}=28\sqrt{2}-4\sqrt{2}=24\sqrt{2}$. Then add 24 + 2 = 26.



3. **21** Apply the Pythagorean theorem on each of the right triangles in the diagram to get CD = 6 and BD = 15 so BC = 21.

4. **75** Let x be the number of girls to remove then $\frac{180-x}{200-x}=\frac{84}{100}\to 18000-100x=16800-84x\to 1200=16x\to x=75$.

5. **1** Since $n=3^x+3^x+3^x=3\cdot 3^x=3^{x+1}$ it follows that $n^2=\left(3^{x+1}\right)^2=3^{2x+2}$. Furthermore since $n^2=9^k=\left(3^2\right)^k=3^{2k}$, we have $3^{2x+2}=3^{2k}\to 2x+2=2k\to x+1=k\to k-x=1$.

6. **35** The prime factorization of 96 is $2^5\cdot 3$. Consider all the possible pairs of factors: 1 and 96, 2 and 48, 3 and 32, 4 and 24, 6 and 16, and finally 8 and 12. Reject the first two and the fourth one since one number is a factor of the other, reject 6 and 16 since 16 is a perfect square, and reject the last one since 8 is a perfect cube. That leaves 3 and 32 whose sum is 35.

7. **85** Start with 256/4 = 64 matches. The 64 winners play 64/4 = 16 more matches. The 16 winners play 16/4 = 4 additional matches. The four winners play one more match for a total of 64 + 16 + 4 + 1 = 85 matches.

8. **7** The y-intercept is 12. Let the x-intercept be k. The legs of the right triangle are 12 and $|k|$ so the third side, the hypotenuse, of the triangle must be 36 – (12 + $|k|$). Apply the Pythagorean theorem to find k: $12^2+k^2=\left(36-\left(12+|k|\right)\right)^2\to 144+k^2=576-48|k|+k^2\to 48|k|=432\to k=\pm 9$. Since the slope is positive k = –9 and the slope is 12/9 = 4/3 and 4 + 3 = 7.

9. **12** Add the two given equations to yield 6a + 6b + 6c = 72. Divide through by 6 to get $a + b + c = 12$.

10. **3** We are given $\frac{a+b}{2}=c, \frac{a+c}{2}=b+1,$ and $\frac{b+c}{2}=2a+2$. These can be rewritten as $a + b - 2c = 0$, $a - 2b + c = 2$, and $-4a + b + c = 4$. Add these three equations together to get $-2a = 6 \to a = -3$ and $|a|=3$.

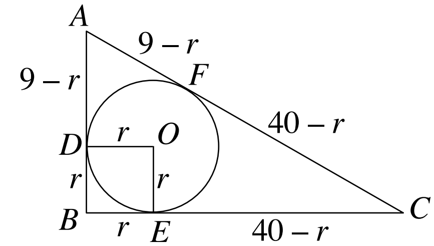
11. **10** The ordered pair (100, 1) can be found by inspection. The other points can be found at $x = 100 - 11k$ and $y = 1 + 20k$, where k = 1, 2, 3, … , 9 which generates nine additional pairs. Thus there are 10 lattice points in the first quadrant.

12. **9** Using a factor tree of prime factors, $6272=2^7\cdot 7^2$. Thus 2 + 7 = 9.

13. **25** Method 1: Factor and solve each equation for $x - y$: $x-y=\frac{75}{x}$ and $x-y=\frac{50}{y}$. Therefore $\frac{75}{x}=\frac{50}{y}$ and $x=\frac{3}{2}y$. Substitute for x in either equation to find that $y = 10$ and then back substitute to find that $x = 15$ so $x + y = 15 + 10 = 25$.

Method 2: Subtract the two given equations to result in $x^2-2xy+y^2=25 \rightarrow (x-y)^2=25$.
Since $x > y$, $x - y = 5$. Add the two original equations to get $x^2-y^2=125 \rightarrow (x-y)(x+5)=125$ and then substitute and solve to get $5(x+y)=125 \rightarrow x+y=25$.

14. **4** Let r be the radius of the circle and D, E, and F be the points of tangency. Using the Pythagorean theorem, we can verify that ΔABC is a right triangle (9-40-41) so $OEBD$ is a square and $BD = OE = OD = BE = r$. Recall that tangent segments drawn to a circle from the same external point are congruent so $AD = AF = 9 - r$ and $CE = CF = 40 - r$. Finally, $(9 - r) + (40 - r) = 41 \rightarrow 2r = 8 \rightarrow r = 4$.



15. **110** Let the numbers be n, $n + 2$, $n + 4$, $n + 6$, and $n + 8$. If we add 10 to the first number, 8 to the second number, 6 to the third number, 4 to the fourth number, and 2 to the fifth number, the five numbers will be equal and divisible by 2, 4, 6, 8, and 10. The smallest number divisible by all five of these numbers is the least common multiple or 120. So $n = 120 - 10 = 110$. Perhaps another way is to just try multiples of 10 that satisfy the remaining four numbers.

Grade Level 10 - NMT 2012 **Solutions**

1. **3** Let the six integers be $n - 6$, $n - 4$, $n - 2$, n, $n + 2$, and $n + 4$. The sum $6n - 6 = -12$ so $n = -1$ and the largest integer is $n + 4 = 3$.

2. **21** Since the diagonals of the quadrilateral are perpendicular we can use the formula $A=\frac{1}{2}d_1d_2$.
Therefore $A=\frac{1}{2}(6)(7)=21$.

3. **256** Let c be the original price then the price after the first reduction is $(3/4)c$ and after four markdowns it would be $(3/4)^4c = 81$. So $81c/256 = 81 \rightarrow c = 256$.

4. **81** Method 1: Let x = acute angle and $90 - x$ = the complement. Then $90-x=\sqrt{x} \rightarrow$
$x^2-180x+8100=x \rightarrow x^2-181x+8100=0 \rightarrow (x-81)(x-100)=0$ so $x = 81$
Method 2: Consider the perfect squares and their complements. The acute perfect square angles are 1, 4, 9, 16, 25, 36, 49, 64, and 81. The only one whose complement is its square root is 81.

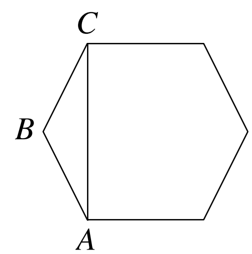
5. **2** Since the lines are perpendicular, their slopes are negative reciprocals. The slope of the first line is –3 and the slope of the second line is $8/a$. Therefore $-3 = -a/8 \rightarrow a = 24$. Substitute 24 into the second equation and isolate y to get $y = (1/3)x + 2$ so the y-intercept is 2.

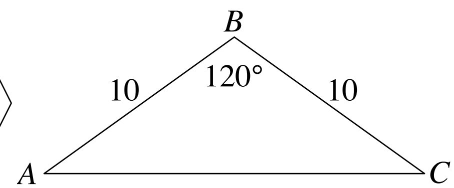
6. **14** Apply the Pythagorean theorem to get $d=\sqrt{8.5^2+11^2}=\sqrt{193.25}=14$ to the nearest integer.

7. **210** For $x > 6$, $3x – 1$ is greater than either $2x$ or $x + 11$. Therefore it must be the measure of the hypotenuse of the triangle. Thus $(3x-1)^2 = (2x)^2 + (x+11)^2 \rightarrow 9x^2 - 6x + 1 = 4x^2 + x^2 + 22x + 121$. It follows that $4x^2 - 28x - 120 = 0 \rightarrow x^2 - 7x - 30 = 0 \rightarrow (x - 10)(x + 3) = 0$. Since $x > 6$, the only value for x is 10. The sides of the triangle are 20, 21, and 29. The area is (.5)(20)(21) = 210.

8. **112** The average roll for a die is 3.5 since three numbers are above and three are below it. Since there are 32 students the most likely total would be 32(3.5) = 112.

9. **300** Let $h = AC$. Rotate ΔABC as in the diagram. The $m\angle B = 120°$ since it is an angle of the hexagon. Draw an altitude from B that bisects $\angle$B since the triangle is isosceles. In a 30°-60°-90° triangle the side opposite the 60° angle is half the hypotenuse times $\sqrt{3}$. So $AC = h = 10\sqrt{3}$ and $h^2 = \left(10\sqrt{3}\right)^2 = 300$.





10. **96** Let n = number of nickels, $2n$ = number of dimes, and $5n$ = number of quarters. Then $(5)n + 10(2n) + 25(5n) = 1800 \rightarrow 150n = 1800 \rightarrow n = 12$, $2n = 24$, and $5n = 60$. So 12 + 24 + 60 = 96.

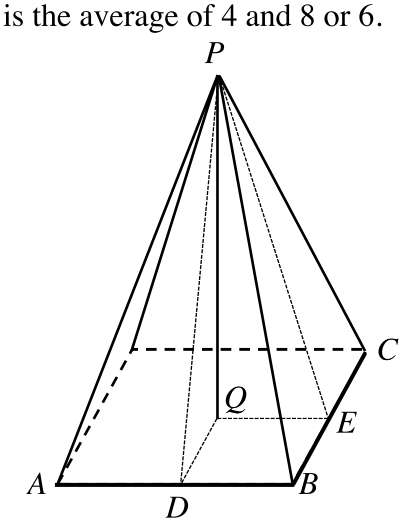
11. **420** The number must have the following factors: 7, 6, 5, and 2. The numbers 1, 3, and 4 will automatically divide the product of those factors and 7·6·5·2 = 420.

12. **24** Isolate the 55 to get $a^2 - b^2 = 55$ so $(a + b)(a – b)$ = either 1×55 or 5×11. If $a + b = 55$ and $a – b = 1$ then $2b = 54$ and $b = 27$. If $a + b = 11$ and $a – b = 5$ then $2b = 6$ and $b = 3$. Then 27 – 3 = 24.

13. **12** Since the area of an equilateral triangle can be found using $A = \frac{s^2\sqrt{3}}{4} \rightarrow 12\sqrt{3} = \frac{s^2\sqrt{3}}{4}$ and $s^2 = 48 \rightarrow s = \sqrt{48} = 4\sqrt{3}$. The perimeter is $P = 3\left(4\sqrt{3}\right) = 12\sqrt{3}$ so $m = 12$.

14. **2** Since the mean is the average of the four numbers, the sum of the four numbers, $3x^2 + 2x + 6 = 4(5.5) = 22 \rightarrow 3x^2 + 2x - 16 = 0 \rightarrow (3x+8)(x-2) = 0$. The only integer solution is $x = 2$ so the numbers are 2, 4, 8, and 8. The mode is 8 and the median is the average of 4 and 8 or 6. The difference between the median and the mode is 2.

15. **564** Let $AB = 18$, $BC = 10$, and $PQ = 12$. Points D and E are midpoints of their respective edges so $\overline{PD}$ and $\overline{PE}$ are altitudes in their respective isosceles triangles. Finally, $QD = 5$ and $QE = 9$. We find that $PD = 13$ and $PE = 15$ using Pythagorean triples, 5-12-13 and 9-12-15, or the Pythagorean theorem. The areas of the front and back faces are .5(18)(13) = 117, the areas of the left and right faces are .5(10)(15) = 75, and the area of the base is 18(10) = 180. The total surface area is 2(117) + 2(75) + 180 = 564.



Grade Level 11 - NMT 2012 **Solutions**

1. **4** When two chords intersect in a circle, the products of the segments of each chord are equal. Therefore, $x(9-x) = 10\cdot 2 \rightarrow x^2-9x+20=0 \rightarrow (x-4)(x-5)=0$ and the smaller value is 4.

2. **4** Make the bases the same so $3^{2x^2-9x} = \frac{1}{81} = 3^{-4}$. Thus $2x^2-9x=-4 \rightarrow 2x^2-9x+4=0$. Factor and set each factor equal to zero, so $(2x-1)(x-4)=0$ and the only integer solution is $x = 4$.

3. **11** If $f(g(x))=x$, then f and g are inverse functions. This implies that $\sqrt{x-2}=3 \rightarrow x = 11$. Thus $f(11) = 3$ and $g(3) = 11$.

4. **17** Secant and cosecant are <u>co</u>mplementary functions. This means that $\sec(3x-20)^\circ = \csc(2x+25)^\circ$ implies that $(3x - 20) + (2x + 25) = 90 \rightarrow x = 17$.

5. **30** When $4x - 12 > 0 \rightarrow x > 3$. Since $x^2-12x<-20 \rightarrow x^2-12x+20>0 \rightarrow (x-10)(x-2)>0$. For this to be true, both factors must have the same sign so $2 < x < 10$. To satisfy these inequalities, $3 < x < 10$ and the product, $a\cdot b$, is $3\cdot 10 = 30$.

6. **2** Recognize that $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ so the denominators are conjugate expressions. Therefore

$$\frac{1}{1+\frac{1}{\sqrt{2}}}\cdot\frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{1}{1^2-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{1-\frac{1}{2}} = 2.$$

7. **301** $\sum_{n=1}^{100}(3n-1) = 2 + 5 + 8 + \ldots + 299$ is an arithmetic series with a difference of 3. Its sum is calculated using the formula $S_n = \frac{n}{2}(a_1+a_n)$ so $S_{100} = \frac{100}{2}(a_1+a_{100}) = 50(2+299) = 50\cdot 301$. Since the problem asked to calculate 1/50 of this sum, the final result is just 301.

8. **8** Let $y=\sqrt{8+x}$ so $\frac{y}{4}=\frac{6}{2+y} \rightarrow y^2+2y=24 \rightarrow y^2+2y-24=0 \rightarrow y = 4$ or $y = -6$. Replace y with $\sqrt{8+x}$ so $\sqrt{8+x}=4 \rightarrow x=8$ or $\sqrt{8+x}=-6$ which is not possible since $\sqrt{8+x} \geq 0$.

9. **14** Convert cotangent to tangent and let $a = \tan(2x)$. Then $\cot(2x)+\tan(2x)=2 \rightarrow \frac{1}{a}+a=2 \rightarrow$ $a^2-2a+1=0 \rightarrow a = 1$ so $\tan(2x) = 1$. When x is on the interval $[0, 2\pi)$, $2x$ is on the interval $[0, 4\pi)$ and $2x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$, and $\frac{13\pi}{4}$. Therefore $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}$, and $\frac{13\pi}{8}$. The sum of these angle measures is $\frac{28}{8}\pi = \frac{7}{2}\pi$ and $7\cdot 2 = 14$.

10. **8** If the first marble chosen is red, the requested probability is $\frac{6}{10}\cdot\frac{4}{9}\cdot\frac{5}{8}\cdot\frac{3}{7}=\frac{1}{14}$. If the first marble chosen is blue, the requested probability is $\frac{4}{10}\cdot\frac{6}{9}\cdot\frac{3}{8}\cdot\frac{5}{7}=\frac{1}{14}$. Thus $\frac{1}{14}+\frac{1}{14}=\frac{1}{7}$ and $1 + 7 = 8$.

11. **400** Let $x = y = 1$. Then $f(2) = f(1 + 1) = f(1) + f(1) = 2\cdot f(1)$.
Similarly $f(3) = f(2 + 1) = f(2) + f(1) = 2\cdot f(1) + f(1) = 3f(1)$
Continue in a similar fashion to find that $f(4) = f(3 + 1) = f(3) + f(1) = 3f(1) + f(1) = 4f(1)$. And ultimately $f(100) = 100f(1) = 100(4) = 400$.

12. **360** Since $\frac{\sin(2x)}{\sin(2y)}=\frac{\sin x}{\sin y} \rightarrow \frac{2\sin x\cos x}{2\sin y\cos y}=\frac{\sin x}{\sin y}$ so $\cos x = \cos y \rightarrow y = 360° - x \rightarrow x + y = 360°$.

13. **125** Use the change of base rule, $\log_b a=\frac{\log_c a}{\log_c b}$, to convert each log expression to base 10 logs.

$$\log_2(x+3)+\log_{\sqrt{2}}(x+3)+\log_{1/2}(x+3)=14 \rightarrow \frac{\log(x+3)}{\log 2}+\frac{\log(x+3)}{.5\log 2}+\frac{\log(x+3)}{-\log 2}=14 \rightarrow$$

$\frac{2\log(x+3)}{\log 2}=14 \rightarrow 2\log(x+3)=14\log 2 \rightarrow \log(x+3)=\log\left(2^7\right)$. Exponentiate base 10 or use the idea that if two log expressions are equal their arguments are equal so $x+3=2^7=128 \rightarrow x = 125$.

14. **13** Without loss of generality, let the coordinates of the vertices be $A(a, 3)$, $B(b, -1)$, and $C(0, 0)$. Since the three sides have equal length, we have by squaring the distance formula,

$$a^2+9=b^2+1=(b-a)^2+16 \rightarrow a^2+9=b^2-2ba+a^2+16 \rightarrow 2ab=b^2+7 \rightarrow a=\frac{b^2+7}{2b}.$$

Substitute this expression to get $\left(\frac{b^2+7}{2b}\right)^2+9=b^2+1 \rightarrow \frac{b^4+14b^2+49}{4b^2}=b^2-8 \rightarrow$

$b^4+14b^2+49=4b^4-32b^2 \rightarrow 0=3b^4-46b^2-49$. This is factorable in terms of b^2 into $\left(3b^2-49\right)\left(b^2+1\right)=0 \rightarrow b^2=\frac{49}{3}$ only. Therefore $b^2+1=\frac{52}{3}=(\text{side})^2$. The area of an equilateral triangle is $A=\frac{s^2\sqrt{3}}{4}=\frac{52}{3}\cdot\frac{\sqrt{3}}{4}=\frac{13\sqrt{3}}{3}$ so $k = 13$.

15. **30** Let $BP = x$, $AD = 4x$, and $AB = CD = y$. Let $(ABCD)$ denote the area of parallelogram $ABCD$. Then $(ABCD) = 4xy\sin C$ and $(DCP) = \frac{1}{2}3xy\sin C=\frac{3}{8}(ABCD)$. Furthermore,

$(ABP) = \frac{1}{2}xy\sin B=\frac{1}{2}xy\sin C=\frac{1}{8}(ABCD)$. $(DAP) = (ABCD) - (DCP) - (ABP) = \frac{1}{2}(ABCD)$.

$(RAP) = \frac{1}{4}(DAP)=\frac{1}{8}(ABCD)=30$. Since a rectangle is a special parallelogram we have another approach. Let the parallelogram be a rectangle with sides of 10 and 24 and determine values for each of the regions. So $(ABP) = 30$, $(PCD) = 90$, $(RAD) = 90$, and $(RAP) = 30$.

Grade Level 12 - NMT 2012 Solutions

1. **503** The functions f and g are inverses of each other. Therefore, the composition $f(g(x)) = x$ and $f(g(2012)) = 2012 \rightarrow \frac{2012}{4} = 503$.

2. **15** Factor $x^3 - 2x^2 - cx = x\left(x^2 - 2x + c\right) = x(x-5)(x-k)$.

3. **7** Method 1: Since $f(1) = 10 \rightarrow f(2) = \frac{29}{3} \rightarrow f(3) = \frac{28}{3} \rightarrow f(4) = \frac{27}{3}$ and so on. Notice the input and the numerator add to 31 in each case. The pattern continues so $f(10) = \frac{21}{3} = 7$.

   Method 2: Rewrite the given recursive equation as $f(n+1) = f(n) - \frac{1}{3}$. Then $f(2) = f(1) - \frac{1}{3} \rightarrow$ $f(3) = f(2) - \frac{1}{3} = f(1) - \frac{2}{3}$, $f(4) = f(3) - \frac{1}{3} = f(1) - \frac{3}{3}$ so $f(10) = f(9) - \frac{1}{3} = f(1) - \frac{9}{3} = 10 - 3 = 7$.

4. **1** We know $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f^{(3)}(x) = \sin x$, and $f^{(4)}(x) = \cos x$ which completes the cycle. Thus $f^{(2012)}(x) = \cos x$ and $f^{(2012)}(2012\pi) = \cos(2012\pi) = 1$.

5. **50** Since ΔABO is a right isosceles triangle with legs = 10, the area is .5(10)(10) = 50.

6. **95** Let a be the first term and r be the common ratio, then $a + ar^2 = 65$ and $ar + ar^3 = 97.5$. Factor to get $a\left(1 + r^2\right) = 65 \rightarrow 1 + r^2 = \frac{65}{a}$ and $ar\left(1 + r^2\right) = 97.5 \rightarrow ar = \frac{97.5}{1 + r^2} = \frac{97.5}{65/a} = \frac{3a}{2}$. Therefore $r = \frac{3}{2}$. Use either original equation to find that $a = 20$, so $ar = 30$ and $a + ar + ar^2 = 65 + 30 = 95$.

7. **16** Complete the square to get the equation $(x+2)^2 + (y+3)^2 = 73$. The slope of the line through the center of the circle (–2, –3) and the point (6,)) on the circle is $-\frac{3}{8}$ so the slope of the tangent to the circle is $\frac{8}{3}$. The equation of the tangent is $y = \frac{8}{3}(x - 6)$ so the y-intercept is 16.

8. **600** Apply the chain rule to get $f'(x) = 3\left(2x^2 - 3\right)^2 \cdot (6x)$ and $f'(2) = 3 \cdot 25 \cdot 8 = 600$.

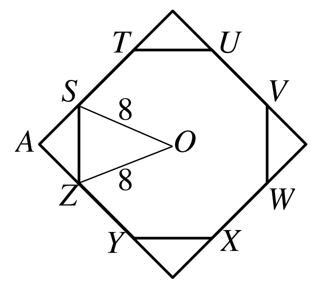
9. **576** The number of factors of $p^r q^s$ is $(r + 1)(s + 1)$ when p and q are prime. This is based upon the idea that in every factor, p may occur anywhere from 0 to r times and q can occur anywhere from 0 to s times. Extending the theorem, we get the number of factors is 3 · 4 · 6 · 8 = 576

10. **540** Substitute $\sin(2x) = 2\sin x \cos x$ and $1 = \sin^2 x + \cos^2 x \rightarrow \sin x + \cos x = \sin^2 x + \cos^2 x + 2\sin x \cos x \rightarrow$ $\sin x + \cos x = (\sin x + \cos x)^2$. Thus $\sin x + \cos x = 1$ or $\sin x + \cos x = 0$. The first equation produces 0° and 90° and the second equation results in 135° and 315°. Finally 0 + 90 + 135 + 315 = 540.

11. **256** Multiply out to get $f(x)=(36-x^2)(x^2-4)$ and the n apply the product rule to find $f'(x)=(36-x^2)\cdot 2x+(x^2-4)\cdot(-2x)=2x(40-2x^2)$. Set $f'(x)=0$ so $x=0$ or $x=\pm\sqrt{20}$. Since $f'(0)$ is negative, the answer is $f'(\pm\sqrt{20})=256$.

12. **0** <u>Method</u> 1 : Apply the expansion $(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+b^3+3ab(a+b)$ with $a=\sqrt[3]{2+\sqrt{x}}$ and $b=\sqrt[3]{2-\sqrt{x}}$ so $a+b=\sqrt[3]{16}$. Cube the given equation: $2+\sqrt{x+2}+2-\sqrt{x}+3\sqrt[3]{4-x}\sqrt[3]{16}=16\rightarrow 3\sqrt[3]{16}\sqrt[3]{4-x}=12\rightarrow\sqrt[3]{16}\sqrt[3]{4-x}=4\rightarrow 16(4-x)=64$. It follows that $x=0$.
<u>Method 2</u>: Realize that $\sqrt[3]{16}=2\sqrt[3]{2}$ so by letting $x=0$ in the given equation we get $\sqrt[3]{2}+\sqrt[3]{2}=2\sqrt[3]{2}$.

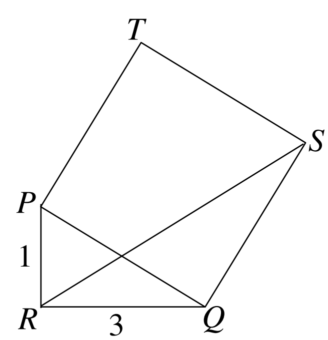
13. **224** In polar coordinates the product of two numbers is the product of their magnitudes and the sum of their angles. Thus $rs=4\cdot 28\sqrt{2}\left(\cos(118°-73°)+i\sin(118°-73°)\right)=112\sqrt{2}cis(45)$ where $cis(x)$ is an abbreviation for ($\cos x + i\sin x$). Thus $rs=112\sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)=112+112=224$.



14. **64** <u>Method</u> 1: Let $\overleftrightarrow{ST}\cap\overleftrightarrow{YZ}=A$. With O as the center of the octagon and radius 8 we find SZ using the Law of Cosines: $SZ^2=8^2+8^2-2(8)(8)\cos 45°$. Thus $SZ^2=128-64\sqrt{2}$. Since ΔASZ is a right isosceles triangle, its area is one-forth of SZ^2 or $32-16\sqrt{2}$. The sum of the areas of the four right triangular regions is $128-64\sqrt{2}$. The area of $\Delta OSZ=\frac{1}{2}(8)(8)\sin 45° = 16\sqrt{2}$. Since there are 8 of these the area of the octagon is $128\sqrt{2}$, so the area of the square is $128-64\sqrt{2}+128\sqrt{2}=128+64\sqrt{2}\rightarrow A=64(2+\sqrt{2})$. It follows that $k=64$.

<u>Method 2</u>: Let $x=SZ$, then the side of the square is $x+2\left(\frac{x\sqrt{2}}{2}\right)=x+x\sqrt{2}=x(1+\sqrt{2})$. The area of the square is $x^2(1+\sqrt{2})^2=x^2(3+2\sqrt{2})$. Since we found in method 1 that $SZ^2=128-64\sqrt{2}=x$, we have the area of he square as $(128-64\sqrt{2})(3+2\sqrt{2})=64(2-\sqrt{2})(3+2\sqrt{2})=64(2+\sqrt{2})$.

15. **5** Use the Pythagorean theorem to find that $PQ=\sqrt{10}=ST$. If we let $\alpha=m\angle QPR$ then $m\angle RQS=90°+m\angle PQR=90°+(90°-\alpha)=180°-\alpha$. Apply the Law of Cosines in ΔRQS to find SR. So $RS^2=3^2+\sqrt{10}^2-2\cdot 3\cdot\sqrt{10}\cos(180°-\alpha) = 19+6\sqrt{10}\cos\alpha$. But the $\cos\alpha=\frac{1}{\sqrt{10}}$ so $RS^2=19+6\sqrt{10}\cdot\frac{1}{\sqrt{10}}=19+6=25$ and $RS=5$.



Mathletics - NMT 2012 **Solutions**

1. **503** Factor 2012 into prime factors: $2012 = 2^2 \cdot 503$. To create a perfect square the least positive multiplier would be 503.

2. **241** \$1 = .67E = .67(64.4R) = .67(64.4(.144Y)) = 6.213312Y so 1Y = \$1/6.213312 and 1500Y = \$1500/6.213312 = \$241.41 = 241 to the nearest dollar.

3. **22** The median is the middle score when there are an odd number of scores. The current median is 85. If both new test grades are below the median, the new median will be in the interval [75, 85). If both test score are above the median then the new median will be in the interval (85, 96]. If one score is above 85 and the other below 85, then 85 remains the median.

4. **241** Since the perimeter of $\Delta ABC = 45$ each side is 15 and the area is $\frac{15^2\sqrt{3}}{4}$. The area of the triangle connecting the midpoints of each side of an equilateral triangle is 1/4 the area of the original triangle so the area of $\Delta PQR = \left(\frac{1}{4}\right)\left(\frac{225\sqrt{3}}{4}\right) = \frac{225\sqrt{3}}{16}$ and 225 + 16 = 241.

5. **25** Sketch the graphs of the functions $y_1 = \left|x^3 + 2x^2 + 10x - 1\right|$ and $y_2 = 2012$ on the same set of axes. Find the approximate points of intersection to be (–13.05, 2012) and (11.74, 2012). For all values in the interval [–13, 11], $y_1 < y_2$. From –13 to 11 is 25 values [11 – (–13) + 1].

6. **184** The n^{th} term of an arithmetic sequence can be found using the formula $a_n = a_1 + (n-1)d$ where a_1 is the first term, n is the number of terms, and d is the difference between consecutive terms. Therefore $7 + (n - 1)(11) > 2012 \rightarrow 11n - 4 > 2012 \rightarrow n > 183.2727$ so $n = 184$.

7. **300** Let r be the radius of the circle. Apply the theorem that tangent segments drawn from a point to a circle are congruent to label the segments that form the sides of the triangle.
<u>Method 1</u>: Apply the Pythagorean theorem to solve for r: $(10-r)^2 + (30+r)^2 = 40^2 \rightarrow$
$2r^2 + 80r - 600 = 0 \rightarrow r^2 + 40r - 300 = 0 \rightarrow r = -20 + 10\sqrt{7} \approx 6.4575$.
Then substitute into the formula for the area of a triangle $A = \frac{1}{2}(bh) = \frac{1}{2}(10+r)(30+r) \rightarrow$
$A = \frac{1}{2}\left(-10+10\sqrt{7}\right)\left(10+10\sqrt{7}\right) = 300$.
<u>Method 2</u>: Since the area of the triangle is $A = \frac{1}{2}(10+r)(30+r) = \frac{r^2+40r+300}{2}$ and $r^2 + 40r - 300 = 0 \rightarrow r^2 + 40r = 300$, we put these together to get $A = \frac{300+300}{2} = 300$. This procedure avoids finding the value of r.

8. **56** <u>Method 1</u>: Let the equation of the parabola be $y = a(x-h)^2 + k$ where (h, k) is the vertex. Since (0, 12) and (20, 12) are points on the parabola, the vertex has an x-coordinate of 10 and the equation becomes $y = a(x-10)^2 + k$. Substitute the coordinates of two points into this equation to get 12 = $100a + k$ and $-7 = 81a + k$. Subtract these two equations to find $a = 1$ and $k = -88$. The quadratic

equation that contains the three original points is $y=(x-10)^2-88$. Since $y = 12x - 100$, solve this system of equations to get the point of intersection in the fourth quadrant to be (4, –52) and 4 – (–52) = 56.
Method 2: Use a quadratic power regression to find the quadratic equation that passes through the three given points and then solve the system as in method 1.

9. **211** The only possible sums not greater than 5 are 3, 4, and 5. There is only one way to get a sum of 3 (1, 1, 1), three ways to get a sum of 4 (1, 1, 2), (1, 2, 1), and (2, 1, 1). There are six ways to get a sum of 5, three permutations of (1, 1, 3) and three permutations of (1, 2, 2). Thus there are 10 rolls that do not satisfy the conditions. All together there are $6^3 = 216$ cases. Thus the probability of rolling a sum greater than 5 is 206/216 = 103/108 and 103 + 108 = 211.

10. **125** The quadrilateral *PQRS* is a square whose area is 1/5 the area of square *ABCD*. This can be shown in several ways.
Method 1: Place the diagram on a coordinate system with *A*(0, 25), *B*(25, 25), *C*(25, 0), and *D*(0, 0). The equation of $\overline{JD}$ is $y = 2x$ and the equation of $\overline{AK}$ is $y = -x/2 + 25$. The intersection of these two lines, *P*, has coordinates (10, 20). Using symmetry we can see that *Q*(20, 15), *R*(15, 5), and *S*(5, 10). Since the slopes of the lines are negative reciprocals we know that $\angle SPQ$ is a right angle and since $PQ = QR = \sqrt{125}$ we know that *PQRS* is a square with area = $\left(\sqrt{125}\right)^2 = 125$.
Method 2: Rotate ΔAPJ 180° about point *P* to form square $BQPP' \cong PQRS$. Repeat on each side of square *ABCD* to see that the area of *PQRS* is one-fifth the area of *ABCD*.

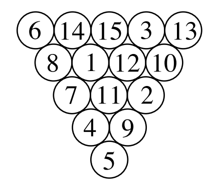
Team Problem Solving - NMT 2012 **Solutions**

1. **747** The only place to start this puzzle is with 2 Down (314). In 5 Across we know that the third box must be 1. Since 3 Down is a power of two it must be 512. Now 1 and 6 Across become obvious followed by 4 Down and 1 Down. The across numbers are: 1357, 9119, 8421 and the down numbers are: 198, 314, 512, and 791. Finally, (1357 + 9119 + 8421) – 10(198 + 314 + 512 + 791) = 18897 – 18150 = 747.

2. **15** Either use ${}_6C_2 = 15$ to determine the number of ways to chose 2 people from a group of 6, or consider the first person has to shake the hands of 5 others, the second has only 4 additional handshakes, the third person 3, then 2, then 1 for a total of 15.

3. **34** Let one leg be the base of the triangle. The area of the triangle is one-half the base times the height. This means we want the height to be a maximum so make the other leg the altitude. Thus the triangle is a right triangle and the third side is the hypotenuse. Use the Pythagorean theorem or the properties of the 45°-45°-90° to get the side to be 34. A second solution is apply the area formula $A = \frac{1}{2}ab\sin C$ where a and b are the given sides. Since the maximum that sin C can equal is 1 and that happens when $m\angle C = 90°$, the triangle must be 45°-45°-90°.

4. **30** Start by realizing that the highest numbers, 13, 14, and 15 must be on the top row. To the left of the 3 must be two consecutive integers, either 13 and 14 or 14 and 15. If you select 13 and 14 you soon realize that there is no place for the 11. Thus select 14 and 15 and place the 13 to the right of the 3. That should be enough of a start to help you get to the end arrangement. The last five numbers to be filled in will all occupy the leftmost side of the triangle and need not be arranged in order to find the requested sum, 4 + 5 + 6 + 7 + 8 = 30.



5. **9** The critical values for this inequality are 3 and –3. Examine the three equations determined by the different domains (–∞, –3), [–3, 3], and (3, ∞). They are $3 - x - 2x - 6 > 12$, $3 - x + 2x + 6 > 12$, and $x - 3 + 2x + 6 > 12$ with solutions sets $x < -5$, $x > 3$, and $x > 3$ respectively. The only integer solutions in their domains are 4, 5, ±6, ±7, ±8, ±9, and ±10. The sum is 4 + 5 = 9.

6. **4** Add the two given equations together to get $6a + 6b + 6c = 24 \rightarrow a + b + c = 4$.

7. **14** Since 4π is about 12.56 and $5\sqrt{2}$ is approximately 7.07, the third side must be greater than the positive difference but less than the positive sum of the given sides. This is based upon the idea that two sides of a triangle must always sum to a number greater than the third side. Thus the third side must be greater than 5.49 but less than 19.63. There are 13 integers satisfying this condition.

8. **12** A test for divisibility by 99 combines the divisibility tests for 9 and 11. The test for 9 is to sum the digits. If the sum of the digits is a multiple of 9 then so is the original number. The test for divisibility by 11 involves adding every other digit and subtracting the sum of the ones that were omitted. If that difference is a multiple of 11 then so is the original number. Therefore $1 + 2 + 3 + 4 + 5 + 6 + 7 + A + B = 9k$ and $(1 + 3 + 5 + 7 + B) - (2 + 4 + 6 + A) = 11n$ for integers k and n. Simplifying we get $28 + A + B = 9k \rightarrow A + B = 8$ or 17 and $4 + B - A = 11n \rightarrow B - A = -4$ or 7. If $A + B = 8$ and $B - A = -4$ we get the solution $B = 2$ and $A = 6$ so the product is 12. The other three arrangements lead to impossibilities.

9. **30** There are 10 vertices in a pentagonal prism. Each vertex has 3 edges connecting to other vertices. Excluding these and the vertex itself, each vertex can connect to 6 additional vertices and each forms an interior diagonal, a base diagonal, or a face diagonal and $10 \cdot 6 = 60$. But each diagonal is counted twice, once from each vertex it connects so the final result is 60/2 = 30. If you count the number of diagonals by type, the results are: 5 in each base, 2 in each face, and 10 in space. The total number is 5(2) + 2(5) + 10 = 30.

10. **2040** To be divisible by the given integers, the number must be a multiple of 60. The first multiple of 60 greater than 2012 can be found by dividing 2012 by 60, rounding the answer up, and then multiplying by 60. 2012/60 = 33.53 and $60 \cdot 34 = 2040$.

11. **48** You could solve three equations in three variables by substituting the given points into the given equation. The third point instantly gives the value of c to be –16. The other two equations are $16a-4b-16=0$ and $64a-8b-16=0$. Alternatively, since the roots are given you could work with the equation in the form $y=a(x+4)(x+8)$. Substitute the point (0, –16) and solve for a. So $a=\frac{-1}{2}$. Multiply out or use the fact that the opposite of the sum of the roots = $-b/a$ = 12, so b = –6. The product of the roots is c/a = 32, so c = –16. Then $|a\cdot b\cdot c| = \left|\left(\frac{-1}{2}\right)(-6)(-16)\right| = 48$.

12. **64** The angles of the rhombus are 60° and 120° since the shorter diagonal is the radius, r, of the circle as well as the sides of the rhombus. The other diagonal is $r\sqrt{3}$ and the area of a rhombus is $A=.5d_1d_2 \rightarrow 32\sqrt{3}=.5r^2\sqrt{3} \rightarrow r^2=64$. The area of the circle is πr^2 so k = 64. Instead of applying the formula for the area of the rhombus you could double the formula for the area of an equilateral triangle $A_{\text{rhombus}} = 2\cdot\frac{s^2\sqrt{3}}{4} = \frac{s^2\sqrt{3}}{2} = 32\sqrt{3} \rightarrow r^2 = s^2 = 64$.

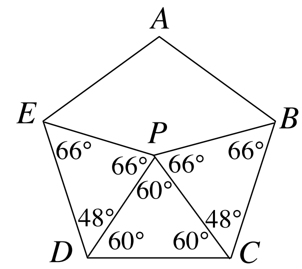
13. **6** Convert the numbers to base ten: $(a+2)(a^2+3a+1)=2a^3+a+2 \rightarrow a^3+5a^2+7a+2=2a^3+a+2$ $\rightarrow a^3-5a^2-6a=0 \rightarrow a(a-6)(a+1)=0$ so the only positive integer base is a = 6.

14. **16** Since all sides of a square are congruent, $\sqrt{32-x^2}=\sqrt{4-3x} \rightarrow 32-x^2=4-3x \rightarrow x^2-3x-28=0$ $\rightarrow (x-7)(x+4)=0$ so x = 7 or –4. We must reject x = 7 since that makes the radicands negative. Thus x = –4 and the sides are $\sqrt{32-(-4)^2}=\sqrt{16}=4$ so the perimeter is 16.

15. **8** Use the quadratic formula to get $x=\frac{3\pm\sqrt{9+8m}}{2m}$. Then $\frac{3+\sqrt{9+8m}}{2m}-\frac{3-\sqrt{9+8m}}{2m}=1 \rightarrow$ $\frac{2\sqrt{9+8m}}{2m}=1 \rightarrow \sqrt{9+8m}=m \rightarrow m^2=9+8m \rightarrow m^2-8m-9=0$. The sum of the two values for m is 8. You could actually find the roots at this point. They are m = –1 or 9 so their sum is 8.

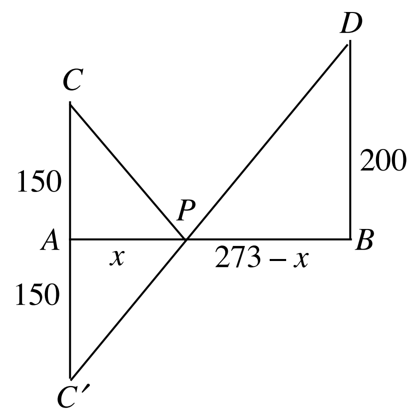
16. **4** Divide 35.21 by 5.03 to get 7, the ratio of consecutive terms. Then solve for n in the sum equation for a geometric series $S=\frac{a(r^n-1)}{r-1}=\frac{5.03(7^n-1)}{7-1}=2012 \rightarrow 7^n=2401 \rightarrow n=\frac{\log 2401}{\log 7} \rightarrow n=4$.

17. **168** Since the three angles of ΔDPC each measure 60° and the interior angles of a regular pentagon each measure 108°, $m\angle PDE = 48°$. Also, since $PD = DC = DE$, ΔDPE is isosceles and $\angle DEP \cong \angle DPE$. Since the sum of the measures in a triangle is 180°, 180° – 48° = 132°, so each angle must measure 66°. Similarly $m\angle BPC = 66°$. The sum of the measures of the angles surrounding point P is 360° so $m\angle EPB = 360° - (60° + 66° + 66°) = 168°$



18. **10** There are 55 cards in the deck. The number of ways to select 2 out of 55 can be found using ${}_{55}C_2$ = (55 · 54)/2 = 1485. Calculate the number of ways of choosing two of each number:
2 ones = not possible, 2 twos, 1 way, 2 threes = ${}_3C_2$ = 3 ways, 2 fours = ${}_4C_2$ = 6 ways, 2 fives = ${}_5C_2$ = 10 ways, etc. These results are known as the triangular numbers and continue as 15, 21, 28, 36, and 45. Either add all of these numbers together or use the formula for the sum of the triangular numbers $S = \frac{n(n+1)(n+2)}{6} = \frac{9(10)(11)}{6} = 165$. Finally, $P(2 \text{ of the same number}) = \frac{165}{1485} = \frac{1}{9}$ and 1 + 9 = 10.

19. **117** Reflect the shorter mast, $\overline{CA}$, through the deck line $\overline{AB}$, to create $\overline{AC'}$. Connect C' to the top of the longer mast, D. The intersection of $\overline{AB}$ and $\overline{DC'}$ is point P the point to connect the guy wires. Since $\Delta APC' \sim \Delta BPD$ all that is left to do is to set up and solve the proportion, $\frac{x}{150} = \frac{273-x}{200} \rightarrow$
$4x = 3(273 - x) \rightarrow x = 117$.



20. **2** Replace all of the functions with sin and cos expressions. So $\frac{\cot^3 x \cdot \csc^5 x}{\sin x \cdot \tan x \cdot \sec^2 x} = 16\cot^8 x \rightarrow$

$\frac{\cos^3 x \cdot \cos x \cdot \cos^2 x}{\sin^3 x \cdot \sin x \cdot \sin x \cdot \sin^5 x} = \frac{16\cos^8 x}{\sin^8 x} \rightarrow \frac{1}{\cos^2 x \cdot \sin^2 x} = 16 \rightarrow$

$(\cos x \cdot \sin x)^2 = \frac{1}{16} \rightarrow \cos x \cdot \sin x = \pm\frac{1}{4} \rightarrow 2\sin x \cdot \cos x = \pm\frac{1}{2}$. But $2\sin x \cdot \cos x = \sin(2x) = \pm\frac{1}{2}$, so $2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Thus $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}$, and $\frac{11\pi}{12}$. The sum of the roots is 2π, so $n = 2$.

Tie Breakers - NMT 2012 **Solutions**

1.

2.

3.