Grade Level 9 - NMT 2011 **Solutions**

1. **50** $\sqrt{8}+\sqrt{18}=2\sqrt{2}+3\sqrt{2}=5\sqrt{2}=\sqrt{50}$.

2. **5** Method 1: Since one root is $\frac{3}{2}$ we can substitute this into the equation to solve for k.

$2\left(\frac{3}{2}\right)^2-13\left(\frac{3}{2}\right)+k=0$. So k = 15. You can now factor and solve for the two roots

$(2x-3)(x-5)=0\rightarrow x=\frac{3}{2}, 5$.

Method 2: Use the idea that the sum of the roots = $-\frac{b}{a}=\frac{13}{2}=\frac{3}{2}+r$, we find that r = 5.

3. **9** The area of a triangle formula is $A=\frac{1}{2}b\cdot h$. Thus $30=\frac{1}{2}\cdot 3\sqrt{5}\cdot h\rightarrow h=\frac{20}{\sqrt{5}}=4\sqrt{5}$ and 4 + 5 = 9.

4. **5** $\frac{\text{Area}}{\text{Circumference}}=\frac{\pi r^2}{2\pi r}=\frac{r}{2}=\frac{5}{2}\rightarrow r=5$.

5. **16** The prime factorization of 2010 is 2·3·5·67. Since each of the primes can either appear once or not at all in each divisor, there are $2^4=16$ possible divisors.

6. **14** Since the numbers on the dice must be different, the first die can show any number, the second die any of the five remaining numbers and the third die any of the four remaining numbers. Therefore, $P(\text{all different})=\frac{6}{6}\cdot\frac{5}{6}\cdot\frac{4}{6}=\frac{5}{9}$ and 5 + 9 = 14.

7. **2** The area of the shaded region can be found by subtracting the area sum of the seven small circles from the area of the larger circle. The area of each small circle is π and the area of the larger circle is 9π so the area of the shaded part is 9π – 7π = 2π, so x = 2.

8. **50** Convert 1 mile in 40 seconds to 1.5 miles per minute. It follows that

$\text{percent increase}=\frac{\text{change}}{\text{original}}=\frac{1.5-1}{1}=.5=50\%$.

9. **6** Let s = the number of Peter's sisters and 2s be the number of his brothers. Let t = the number of Anna's sisters and 5t is the number of her brothers. Since the total number of girls in the family is both s and t + 1, we get s = t + 1. Similarly, the number of boys in the family is both 2s + 1 and 5t, we get 2s + 1 = 5t. solve these using substitution to arrive at s = 2 and t = 1.

10. **100** Factor to get $\frac{(200)^2\times(700)^2}{(100x)^2}=14^2\rightarrow\frac{2^2(100)^2\times 7^2(100)^2}{x^2(100)^2}=14^2\rightarrow\frac{2^2\times 7^2\times 100^2}{x^2}=14^2$.

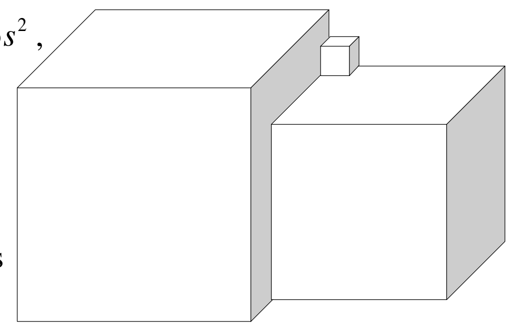
Therefore $x^2=\frac{2^2\times 7^2\times 100^2}{14^2}\rightarrow x=100$.

11. **202** Let c = cost to dealer of one car and Let d = the cost to the dealer of the second car. Then $(1 + .1)c = 9999$ and $(1 - .1)d = 9999$. Thus $c = \$9090$ and $d = \$11110$. Therefore the dealer lost $(11110 + 9090) - 2(9999) = 20200 - 19998 = 202$.

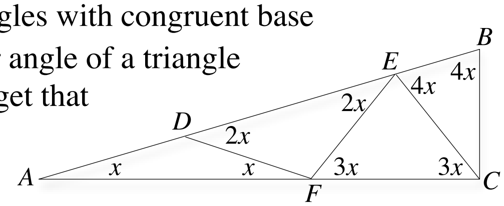
12. **705** The sequence of odd integers that have a remainder of 2 when divided by 3 are: 5, 11, 17, 23, … which can be generated by the expression $6n - 1$. Therefore, the fifteenth number in the sequence is $6(15) - 1 = 89$. To add the integers 5, 11, 17, 23, …, 83, 89 in an easy way, notice that $5 + 89 = 11 + 83 = 17 + 77 = \ldots = 94$. Hence there are 7 pairs that add to 94 and the middle number, 47, which has no partner. The sum is $7(94) + 47 = 705$.

13. **30** Since m is the number of students in each group, $5m$ is the number of students in the class. When 6 students are absent there are 6 groups of $m - 2$ or $6(m - 2)$ students. Therefore $5m - 6 = 6(m - 2)$. Solving we get $m = 6$ and $5m = 30$.

14. **536** The surface area of each cube can be found using $SA = 6s^2$, so the areas are 24 sq cm, 216 sq cm, and 384 sq cm respectively. Gluing the 6 cm and 8 cm cube we lose 36 sq cm from each cube. Gluing the 6 cm and the 2 cm together we lose 4 sq cm from each and similarly gluing the 2 cm and the 8 cm cubes together we lose 4 sq cm from each. Thus the total surface are of the final object is $24 + 216 + 384 - 2(36 + 4 + 4) = 536$.



15. **18** ΔADF, ΔDFE, ΔFEC, and ΔECB are all isosceles triangles with congruent base angles. Let $m\angle A = x$, then by recalling that the exterior angle of a triangle equals the sum of the non-adjacent interior angles, we get that $m\angle FDE = 2x$, $m\angle EFC = 3x$, and $m\angle BEC = 4x$. Since $x + 4x = 90$, we find that $x = 18$.



Grade Level 10 - NMT 2011 **Solutions**

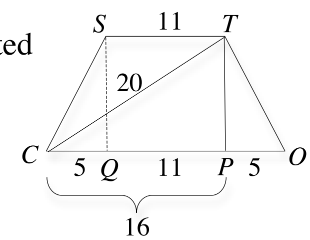
1. **0** Let the five numbers be represented by $n - 2, n - 1, n, n + 1, n + 2$. The sum of these five expressions is $5n = -10$, so $n = -2$ and the largest value is $-2 + 2 = 0$.

2. **6** Set the slope of $\overleftrightarrow{AB}$ = slope of $\overleftrightarrow{BC}$ to get $\frac{-2-14}{18-(-6)} = \frac{k-(-2)}{k-18} \rightarrow \frac{-16}{24} = \frac{-2}{3} = \frac{k+2}{k-18}$. Solve for k by multiplying the equation by the LCD to get $-2k + 36 = 3k + 6 \rightarrow k = 6$.

3. **30** Let x be the angle, then $90 - x$ is the complement and $180 - x$ is the supplement. Solve the equation $180 - x = 2.50(90 - x) \rightarrow 180 - x = 225 - 2.5x \rightarrow 1.5x = 45 \rightarrow x = 30$.

4. **200** The volume according to Joan is $3 \cdot 3 \cdot 22 = 198$. Since the actual volume, x, was greater than her calculation and her relative error was 1%, we solve the equation $.99x = 198$, so $x = 200$.

5. **192** Examine the diagram at the right. Altitudes $\overline{SQ}$ and $\overline{TP}$ can be calculated using the Pythagorean theorem or by recognizing the appropriate multiple of the Pythagorean triple 3-4-5. The result is 12. Therefore the area of the trapezoid is $A = \frac{h}{2}(b_1 + b_2) = \frac{12}{2}(11+21) = 192$.



6. **39** $f(p) = p + 3 = 15 \rightarrow p = 12$. Thus $f(3p) = f(36) = 36 + 3 = 39$.

7. **81** Since $\frac{x-y}{x+y} = \frac{4}{5}$ we can multiply both sides of the equation by $5(x + y)$ to get $x = 9y$. It follows that $\frac{x^2}{y^2} = \frac{(9y)^2}{y^2} = 81$.

8. **8** The slope of the given line segment is $m = \frac{3-1}{5-1} = \frac{1}{2}$. Therefore the slope of the perpendicular bisector will be -2, the opposite reciprocal. The coordinates of the midpoint of the given segment are (3, 2). The equation of the perpendicular bisector is $y - 2 = -2(x - 3)$ and its y-intercept can be found by replacing x with 0 to get $y = 8$.

9. **12** The diagonals of a rectangle are congruent so the question is asking for the square of one of its diagonals. The diagonal can be calculated using the Pythagorean theorem $d = \sqrt{3^2 + \left(\sqrt{3}\right)^2} = \sqrt{12}$ so $d^2 = 12$.

10. **135** The ratio of the areas of two similar polygons is equal to the ratio of the squares of a pair of corresponding linear measurements. This means $\frac{A_1}{A_2} = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$. Since the area of the original figure is 240, we get $\frac{A_1}{240} = \frac{9}{16} \rightarrow A_1 = \frac{9 \cdot 240}{16} = 135$.

11. **72** The average speed of the train can be found by dividing the total distance by the total time. If we let d be the one-way distance then the time going is $d/60$ and the time returning is $d/90$. Hence

$$\text{Average Speed} = \frac{2d}{\frac{d}{60}+\frac{d}{90}} = \frac{2d\cdot 180}{3d+2d} = \frac{360}{5} = 72 .$$

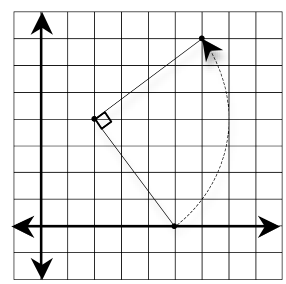
12. **384** Let x be the measure of the unknown edge of the prism so $2x$ will be the measure of the edge of the cube. Solve for x in the equation $2x^2+4(22x)=6(2x)^2 \rightarrow 2x^2+88x=24x^2$. Since x cannot be zero, we get $22x = 88$ and $x = 4$. The surface area of the cube is $24 \cdot 16 = 384$.

13. **25** The height of the trapezoid can be calculated based upon the fact that the angles of the octagon are all 135°, so when the altitudes are drawn the two small triangles formed are isosceles right triangles. The measure of each leg is $\frac{1}{2}\cdot 10\cdot\sqrt{2}$. Therefore the area is

$A=\frac{h}{2}(b_1+b_2)=\frac{5\sqrt{2}}{2}\left(10+\left(10+10\sqrt{2}\right)\right)=\frac{5\sqrt{2}}{2}\left(20+10\sqrt{2}\right)=5\sqrt{2}\left(10+5\sqrt{2}\right)=50+50\sqrt{2}$. The area of the octagon can be found using $A_{regular\ \ polygon}=\frac{1}{2}ap=\frac{1}{2}\cdot\left(5+5\sqrt{2}\right)\cdot 80=200+200\sqrt{2}$.

Thus the ratio of the area of the trapezoid to that of the octagon is $\frac{1}{4}=25\%$.

14. **42** <u>Method 1</u>: Translate both points so that the center of rotation is (0, 0). This means that the point (5, 0) moves to (3, –4). The rotation of 90° counterclockwise moves this point to (4, 3). Now translate this point back based upon the initial translation (–2, –4). The result is ((6, 7). The product of these coordinates is 42.
<u>Method 2</u>: Draw the diagram and count the boxes.



15. **5** Add the 4 expressions, divide by 4 and set equal to 5. This results in the following equation: $\frac{x}{3}+(x+1)+2x+\frac{27}{x}=20 \rightarrow 10x^2+3x+81=60x \rightarrow 10x^2-57x+81=0$. Factor and solve: $(10x-27)(x-3)=0 \rightarrow x=3$ or $x=\frac{27}{10}$. Reject the fractional answer. The 4 numbers are: 1, 4, 6, and 9 so the median is midway between 4 and 6 which is 5.

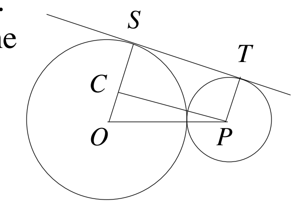
Grade Level 11 - NMT 2011 **Solutions**

1. **512** Consider a right triangle in which the hypotenuse is the main diagonal of the cube and each leg is a side of the cube, with length s. Thus, a face diagonal of the cube has length $s\sqrt{2}$. The Pythagorean theorem yields $s^2+\left(s\sqrt{2}\right)^2=\left(8\sqrt{3}\right)^2 \rightarrow 3s^2=192 \rightarrow s=8$ and the volume is $8^3=512$.

2. **3** The product of the roots is $\frac{c}{a}=-\frac{5}{3}$ and the sum of the roots is $\frac{-b}{a}=\frac{-14}{3}$. The product exceeds the sum by $-\frac{5}{3}-\left(\frac{-14}{3}\right)=\frac{9}{3}=3$.

3. **3** When Carole runs 10 miles, Diane runs 8 miles. At the same rates, when Carole runs 15 miles, Diane runs 12 miles. Therefore, Carole can beat Diane by 3 miles in a 15-mile race.

4. **24** Call the center of the larger circle O and the center of the smaller circle P. Let the common external tangent line intersect the larger circle at S and the smaller circle at T. Since a radius drawn to the point of tangency of a tangent line is perpendicular to the tangent, $\angle OST$ and $\angle PTS$ are right angles. From P drop a perpendicular to $\overline{OS}$ and label the intersection C. $ST = PC$, $OC = 10$ and $OP = 26$. Therefore, in right ΔOPC, $PC = 24$.

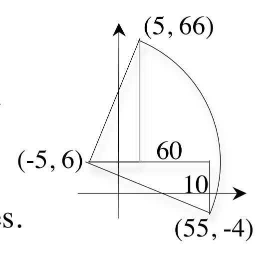


5. **78** Calculate $a_2=50+4=54$, $a_3=54+6=60$, $a_4=60+8=68$, $a_5=68+10=78$.

6. **101** Reduce each fraction and group in twos to create the following sequence (2 – 3) + (4 – 5) + (6 – 7) + … + (198 – 199) + 200 = –1(99) + 200 = 101.

7. **20** Let the diagonals of rhombus $ABCD$ intersect at E. The length of the altitude of ΔABE is the length of the radius of the inscribed circle. The area of ΔABE can be calculated using both legs of the right triangle or by using base $\overline{AB}$ and the height, the radius of the inscribed circle. Therefore $\frac{1}{2}(5)(10)=\frac{1}{2}\left(5\sqrt{5}\right)r \rightarrow r=2\sqrt{5}$ and $A=\pi r^2=20\pi$. Note: This problem can be done using the idea that the radius drawn to the base of the rhombus is the altitude to the hypotenuse of the triangle making it the mean proportional between the segments of the hypotenuse. This can be found using similar triangles.

8. **142** Translate the plane 5 units to the right and down 6 units so that the new coordinates of Q, called Q' are (0, 0). Then P' is (60, –10). After applying a 90° CCW rotation about Q', the image of P' is P''(10, 60). Reversing the initial translation moves P'' left 5 units and up 6 units to (5, 66) and 2(5 + 66) = 142. Alternatively, you could draw a diagram and "count" boxes.



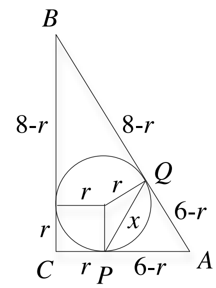
9. **125** Since $81=3^4$, the second equation becomes $y=3^{x-4}$. Since $32=2^5$, the first equation after substituting for y, becomes $2^{x-5}=3^{x-5}$, so $x = 5$ and $y = 3$. Thus $x^y=5^3=125$.

10. **140** Since $\log(5x+130)-\log(8x-2)=\log\left(\dfrac{5x+130}{8x-2}\right)=1$, if we apply the inverse log to both sides we get $\dfrac{5x+130}{8x-2}=10 \rightarrow 5x+130=80x-20 \rightarrow x=2$. So $5x + 130 = 140$.

11. **200** The critical values for the given equation are found by setting each absolute value expression equal to zero. Therefore the critical values are 1/2 and 2. When $x < 1/2$, solve $1 - 2x + 2 - x = 4$, so $x = -1/3$. When $1/2 < x < 2$, solve $2x - 1 + 2 - x = 4$. The solution $x = 3$ must be rejected. When $x > 2$, solve $2x - 1 + x - 2 = 4$. The solution is $x = 7/3$. Thus $100\left(\dfrac{7}{3}+\dfrac{-1}{3}\right)=200$.

12. **48** Use trig identities to get $\left(\dfrac{1}{\sin x}+\dfrac{\cos x}{\sin x}\right)(1-\cos x)=\dfrac{1-\cos^2 x}{\sin x}=\dfrac{\sin^2 x}{\sin x}=\sin x$. Since $\sin x=\cos 42°$, $x = 48°$.

13. **8** Segments from a point, tangent to a circle are congruent. Using this theorem, we can solve for the radius of the inscribed circle (see diagram). Since $AQ + BQ = 10$, $8 - r + 6 - r = 10$, so $r = 2$. Apply the Law of cosines to ΔAQP to get $x^2 = 4^2+4^2-2(4)(4)\cos A$. Then substitute, $\cos A=\dfrac{\text{adjacent}}{\text{hypotenuse}}=\dfrac{6}{10}$, so $x^2=32-32\left(\dfrac{6}{10}\right)=\dfrac{64}{5}\rightarrow x=\dfrac{8}{\sqrt{5}}$ and $k = 8$.



14. **10** Since $f^{-1}(3)=x$, $f(x)=3=\dfrac{2x+1}{3x-23}$. Cross multiplying yields $2x + 1 = 3(3x - 23)$ so $2x + 1 = 9x - 69$ and $x = 10$.

15. **77** There are three distinct sets of integers less than 7, whose product is 24. They are {1, 4, 6}, {2, 3, 4}, and {2, 2, 6}. There are six permutations for the first and second sets and only three arrangements for the third set. Therefore the requested probability is $\dfrac{6+6+3}{6^3}=\dfrac{15}{216}=\dfrac{5}{72}$. The required sum is $5 + 72 = 77$.

1. **685** Apply $a_n = a_1 + (n-1)d$, so $a_{100} = -8 + (100-1)7 = 685$.

2. **168** Use the chain rule: $f'(x) = 2(x^3-1)(3x^2) \to f'(2) = 2(7)(12) = 168$.

3. **125** Since $g(f(x)) = g(5x-4) = a(5x-4)+b = 50x+75$, we can see that $5a = 50$ and $-4a + b = 75$. It follows that $a = 10$ and $b = 115$, so $a + b = 125$.

4. **20** The left side of the equation is a sum of reciprocals and the right side may also be expressed as a sum of reciprocals, $2 + \frac{1}{2}$. Therefore either $\sqrt{\frac{x+10}{x-2}} = 2$ or $\sqrt{\frac{x+10}{x-2}} = \frac{1}{2}$. Square and solve for x to get 6 and -14. The sum of their absolute values is 20.

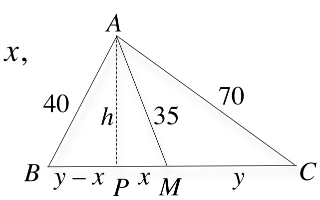
5. **334** Factor to cancel: $\lim\limits_{x\to-3} \frac{x^2+340x+1011}{x+3} = \lim\limits_{x\to-3} \frac{(x-3)(x+337)}{x+3} = \lim\limits_{x\to-3}(x+337) = 334$.

6. **134** By inspection, the smallest value for x yields the ordered pair (2, 401). Since the slope tells us that by increasing x by 5, we must decrease y by 3, we find that the smallest value of y will result in the ordered pair (667, 2). We can then solve $2 + 5n = 667$, so $n = 133$. The number of pairs of integers corresponds to all n from 0 to 133 so there are 134 pairs all together.

7. **55** Since $\log\left(\frac{1}{2^k}\right) = \log\left(\frac{1}{2}\right)^k = k\log\left(\frac{1}{2}\right)$, $\frac{\sum_{k=1}^{10}\log\left(\frac{1}{2^k}\right)}{\log\left(\frac{1}{2}\right)} = \frac{\sum_{k=1}^{10}k\log\left(\frac{1}{2}\right)}{\log\left(\frac{1}{2}\right)} = \frac{\log\left(\frac{1}{2}\right)\cdot\sum_{k=1}^{10}k}{\log\left(\frac{1}{2}\right)} = \sum_{k=1}^{10}k = 55$.

8. **9** Since $f'(10)$ is the slope of $f(x)$ at $x = 10$ and the slope can also be calculated using the two-point form, $\frac{68-23}{15-10} = 9$, we can set these expressions equal to find $f'(10) = 9$.

9. **90** In ΔABC, draw the altitude from A to $\overline{BC}$ and label the base P. Let $PM = x$, $MC = y$, $AP = h$, and $PB = y - x$. Apply the Pythagorean theorem to each of the three right triangles in the diagram and solve for h^2.



$40^2 - (y-x)^2 = 70^2 - (y+x)^2$ and $40^2 - (y-x)^2 = 35^2 - x^2$. Rearrange each equation $(y+x)^2 - (y-x)^2 = 70^2 - 40^2$ and $(y-x)^2 - x^2 = 40^2 - 35^2$. Now factor using the difference of squares and simplify: $((y+x)+(y-x))((y+x)-(y-x)) = (70+40)(70-40) \to$ $(2y)(2x) = 30\cdot110 \to 2xy = 1650$ and $((y-x)+x)((y-x)-x) = (40+35)(40-35) \to$ $y(y-2x) = 375 \to y^2 - 2xy = 375$. Substitute and solve: $y^2 - 2xy = 375 \to y^2 - 1650 = 375 \to$ $y^2 = 2025 \to y = 45$ and $BC = 2y = 90$.

10. **79** Note that $\frac{1}{r^2}+\frac{1}{s^2}=\frac{r^2+s^2}{(rs)^2}$ and that $r^2+s^2=(r+s)^2-2rs$. Since $r+s=\frac{b}{a}=1$ and $rs=\frac{c}{a}=8$, we can substitute these values as follows: $\frac{1}{r^2}+\frac{1}{s^2}=\frac{(r+s)^2-2rs}{(rs)^2}=\frac{1^2-2(8)}{8^2}=\frac{-15}{64}$. The sum $|-15|+|64|=79$.

11. **200** Square both sides of the given equation to get $\sin^2 x+2\sin x\cdot\cos x+\cos^2 x=\frac{1}{41}$. Since $\sin^2 x+\cos^2 x=1$ and $2\sin x\cdot\cos x=\sin(2x)$, we have $\sin(2x)=-\frac{40}{41}$. Use right triangle trig or trig identities to arrive at $\tan(2x)=\frac{40}{9}$ so $45\tan(2x)=200$.

12. **4** The absolute maximum can be fund on the axis of symmetry $\left(x=\frac{-b}{2a}=\frac{-24}{-18}=\frac{4}{3}\right)$ or by the derivative set to zero. Substitute this result into the equation to find y = 16. The minimum is at one of the endpoints (in this case the one furthest from the axis of symmetry). The coordinates of the minimum are (2, 12) The absolute value of the difference of these is 16 – 12 = 4.

13. **82** The probabilities of success on the first question for each of the students are: 0.5, 0.4, 0.25, and 0.2. The probability that they all fail to answer the first question correctly is 0.5, 0.6, 0.75, and 0.8. The product of these values is $\frac{18}{100}$, so the product of at least one correct result is $1-\frac{18}{100}=\frac{82}{100}$.

14. **4** The terms of the sequence are: 15, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, … . Notice that starting with the sixteenth term, the three numbers 4, 2, and 1 repeat endlessly in that order. Since 2011 is one more than a multiple of 3, the term in the sequence will be 4.

15. **20** On the graph of the hyperbola in quadrant I, call the point of tangency *P*(2011, 10/2011). Since the triangle is a right triangle with a right angle at the origin, its area is one-half the product of the *x*- and *y*-intercepts of the line tangent to the hyperbola. With $y'=-\frac{10}{x^2}$, the equation of the tangent line is $y-\frac{10}{2011}=-\frac{10}{2011^2}(x-2011)$. Therefore the intercepts are (4022, 0) and (0, 20/2011).
Thus the area of the triangle is $\frac{1}{2}\cdot\frac{20}{2011}\cdot 4022=20$.

1. **43** The total volume of the room is $V = l \times w \times h = 5 \times 8 \times 8 = 320$ cubic feet. Since we want a fan with enough power to remove the entire volume of air 8 times per hour we multiply by 8 and divide by 60 to get $\text{cfm} = 320 \times 8 \div 60 = 42.67$ which rounds to 43.

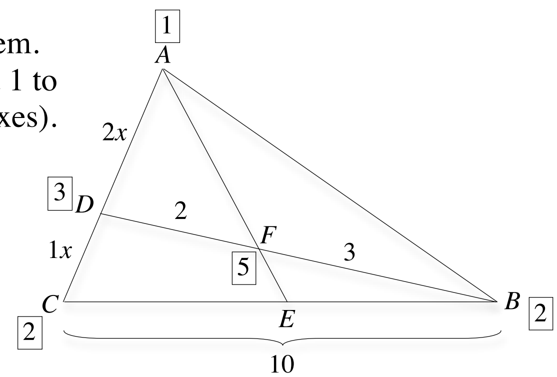
2. **14** Change all of the bases to the same value: $4^x + 2^{2x} + 2^{2x+1} = \dfrac{16}{64^{4x}} \to 2^{2x} + 2^{2x} + 2 \cdot 2^{2x} = \dfrac{2^4}{2^{24x}}$. Simplify as follows: $4 \cdot 2^{2x} = 2^{4-24x} \to 2^{2x+2} = 2^{4-24x} \to 2x + 2 = 4 - 24x$. Solving for x results in $26x = 2 \to x = \dfrac{1}{13}$. Add 1 + 13 = 14.

3. **782** Let the radius of the semicircular arc be r, then the straight part is $2r$ and the curved part is πr. The length of the track is $2r + \pi r = 2011 \to r = \dfrac{2011}{2+\pi} \approx 391$. Therefore the straight part is 782.

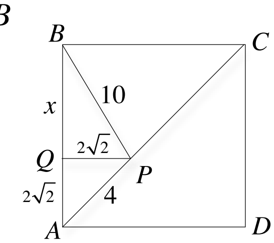
4. **105** The total of the top 10 scores is 10(117.3) = 1173 while the total of the top six scores is 6(125.5) = 753. The difference, 1173 – 753 = 420 is the sum of the four scores whose average we are asked to calculate. Therefore the average is 420/4 = 105.

5. **288** There are 2 flavors of donut, 4 choices of filling (including none), 12 frosting choices, and 3 glazes (including no glaze). By the counting principle we get 2·4·12·3 = 288.

6. **23** Mass point geometry should be used to solve this problem. Since the ratio of CD:DA = 1:2, assign weights of 2 and 1 to points C and A respectively (weights are enclosed in boxes). Then the total weight at D, the fulcrum is 3. Since $CE = EB$ the weight at B must be 2 as well. Finally, since the product of weight times distance is constant along a line segment, the ratio of the distances DF:FB must be 2:3 and therefore 10(2) + 3 = 23.

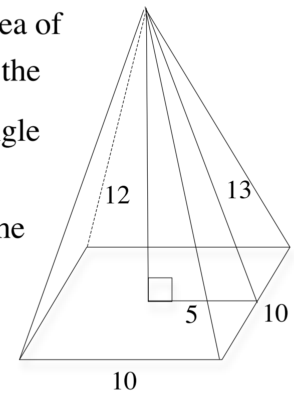


7. **154** Since $m\angle BAP$ is 45°, $\overline{AP}$ must be a part of diagonal $\overline{AC}$. Draw $\overline{PQ} \perp \overline{AB}$ as seen in the diagram. Since ΔAPQ is a 45°-45°-90° triangle, $AQ = QP = 2\sqrt{2}$. Apply the Pythagorean theorem to find that $BQ = \sqrt{10^2 - \left(2\sqrt{2}\right)^2} = \sqrt{100-8} = \sqrt{92}$ and $AB = 2\sqrt{2} + \sqrt{92} \approx 12.42$ so the area of the square $= 12.42^2 = 154.26$ or 154 to the nearest integer.



8. **49** The graph of $|x| + |3y| = 150$ is a diamond shape centered at the origin. The portion in the first quadrant, where all of the coordinates are positive is just a line segment that intersects the axes at (0, 50) and (150, 0). Since the slope is –1/3, the integer ordered pairs will have x-coordinates of 0, 3, 6, 9, …, 150. There are 51 such pairs. Subtract the endpoints that contain 0, we get 49 pairs.

9. **360** The total surface area of the pyramid = $4 \times A_{face\ triangle} + A_{square\ base}$. The area of the base = 100. Since the volume of the pyramid = 1200 and the area of the base is 100, the height of the pyramid is 12 ($V = \frac{1}{3} \times A_{base} \times h$). The triangle formed by the height of the pyramid, the slant height of a face, and the segment connecting the feet of the two segments is a 5-12-13 triangle. The area of each triangular face is $\frac{1}{2} \times \text{base} \times \text{slant height} = \frac{1}{2}(10)(13) = 65$. Thus the total surface area is 4(65) + 100 = 360.



10. **29** Method 1: Select any point on one of the two given lines and create the equation of the line perpendicular to the two lines containing the given point. For example (5, 0) satisfies the first equation so $21x - 20y = 105$ is perpendicular to the given lines and contains (5, 0). Find the intersection of this line with the second equation by solving the system of equations: $21x - 20y = 105$ and $20x + 21y = 941$. Multiply the first equation by 21 and the second by 20 to eliminate the y and solve for x to get the point (25, 21). Lastly, apply the distance formula between the points (5, 0) and (25, 21) to get $d = \sqrt{(25-5)^2 + (21-0)^2} = \sqrt{841} = 29$.

Method 2: The formula used to find the distance from any point (x_1, y_1) to the line whose equation is $ax + by + c = 0$ is $d = \frac{|ax_1 + by_1 + c|}{\sqrt{x_1^2 + y_1^2}}$. In this application we find that the distance can be found using $d = \frac{|20(5) + 21(0) - 941|}{\sqrt{20^2 + 21^2}} = \frac{841}{29} = 29$.

Team Problem Solving - NMT 2011 Solutions

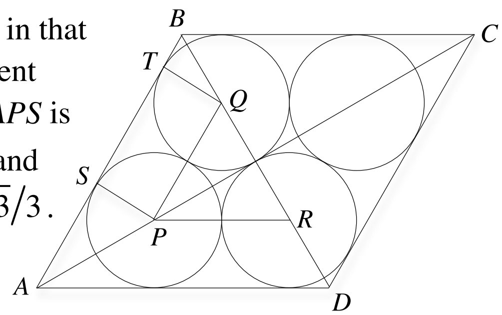
1. **875** The across numbers are: 1313, 2744, 1743 and the down numbers are: 121, 377, 144, and 343.

2. **10** Have the squares overlap at any angle between 0° and 90° then count the regions.

3. **104** Work the problem backwards. Let A, B, and C each end with \$64. If C was the last to lose he had to pay A and B amounts equal to what they had before the last game. Therefore both A and B started the round with \$32 and C must have had \$128. If B was the loser in the second round C and A each must have had \$64 and \$16 respectively. This means that B had to have started the round with 32 + 64 + 16 = \$112. Finally A would have lost in the first round so B and C must have started the process with \$56 and \$32 respectively so A had 16 + 56 + 32 = \$104.

4. **36** The number of divisors of perfect squares are always odd. The number of divisors for 4 is 3, for 9 is 3, for 16 is 5, for 25 is 3, and for 36 is 9 (1, 2, 3, 4, 6, 9, 12, 18, and 36).

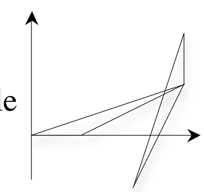
5. **41** The key to this problem is to realize that ΔABD is equilateral in that it is similar to ΔPQR (the line joining the centers of two tangent circles is parallel to the common external tangents). Since ΔAPS is a 30°-60°-90° triangle with $SP = 1$, $AS = \sqrt{3}$; $ST = PQ = 2$; and ΔQBT is also a 30°-60°-90° triangle with $TQ = 1$, so $BT = \sqrt{3}/3$.



Therefore, $BA = AS + ST + TB = \sqrt{3} + 2 + \frac{\sqrt{3}}{3} = 2 + \frac{4\sqrt{3}}{3}$.

The area of a rhombus can be found using a variety of formulas, but the easiest in this case might be $A = ab\sin C = \left(2 + \frac{4\sqrt{3}}{3}\right)\left(2 + \frac{4\sqrt{3}}{3}\right)\sin(60°) = \frac{24 + 14\sqrt{3}}{3}$. Finally 24 + 14 + 3 = 41.

6. **50** Draw the diagram to discover that $\Delta ABD \cong \Delta EDC$. Since the base of the triangle is 10 and the height is also 10, the area is .5(10)(10) = 50.

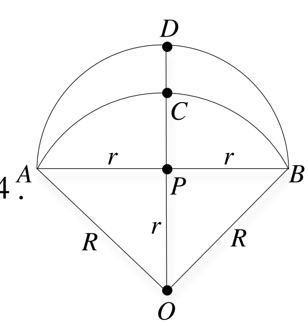


7. **29** Solve by multiplying the first equation by 3 and the second by 7 and then add them together to eliminate the y-term. The resulting equation is $78x^2 = 22542 \rightarrow x = 17$. Substitute back into one of the original equations to find that $y = 12$, so $x + y = 17 + 12 = 29$.

8. **70** Let x = the number of marbles in the first bag, then $3x$ and $6x$ represent the number of marbles in bags two and three. The number of red marbles in each bag can be represented as $x/2$, $2x$, and $9x/2$ respectively. Therefore the percent of red is $\frac{100(x/2 + 2x + 9x/2)}{x + 3x + 6x} = 70$.

9. **34** Let the center of the larger circle be O and the center of smaller be P. The radius of the larger circle, $R = r\sqrt{2}$, where $r = 34 + 17\sqrt{2}$. This is based upon the fact that $\angle AOB$ is a right angle (a central angle = intercepted arc) and that $PA = PO = PB = PD$. Therefore, $R = \left(34 + 17\sqrt{2}\right)\sqrt{2} = 34\sqrt{2} + 34$.

It follows that $DC = 2r - R = 68 + 34\sqrt{2} - \left(34\sqrt{2} + 34\right) = 34$.



10. **740** The only integers with prime factors of 3 or 7 and that do not have a perfect cube factor are powers of 3 and 7 raised to powers 0, 1, or 2. These numbers are 3, 9, 7, 49, 21, 63, 147, and 441. The sum of these integers is 740.

11. **60** Find one number that satisfies the conditions and instead of arranging the numbers linearly, such as 143256, arrange them in a circle as shown in the diagram. From this circle of numbers (the only possible arrangement of digits) count the number of numbers. One can begin at any of the six digits and move either clockwise or counterclockwise about the circle. Therefore there are only 12 such numbers. The reciprocal of 12/720 is 720/12 = 60.

1 4
6 3
5 2

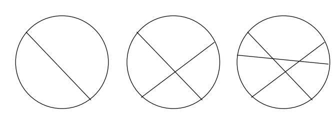
12. **90** Since $NC = NL$ and I is the midpoint of the base of isosceles ΔNCL so $NI \perp CL$. We are given that $MI = ML = NI$ so ΔNIM is a right isosceles triangle and $m\angle INM = 45°$. Let $a = m\angle CNI = m\angle LNI$. It follows that $m\angle LNM = a - 45°$ and $m\angle CNM = a + 45°$. Subtract, $(a + 45) - (a - 45) = 90$.

13. **60** Count the number of each configuration: 1×1 (12), 1×2 (9), 1×3 (6), 1×4 (3), 2×1 (8), 2×2 (6), 2×3 (4), 2×4 (2), 3×1 (4), 3×2 (3), 3×3 (2), 3×4 (1) and then add 12+9+6+3+8+6+4+2+4+3+2+1 = 60. You might note that there is a pattern based upon my grouping (12, 9, 6, 3), (8, 6, 4, 2), and (4, 3, 2, 1) so that counting the number of rectangles in a larger array would not require as much work.

14. **196** The easiest way to handle this problem is to rotate the triangle about the vertex of the right angle so that the congruent sides pass through two of the corners of the square and the overlap becomes one-quarter of the square ($A = 28^2/4 = 196$). It is interesting to note that the overlap under this type of rotation is constant. The proof is based upon congruent triangles and is left to the reader.

15. **27** Each of the three missing numbers must be a 3 so their product is 27. In order for $P(C \text{ beat } D) = 2/3$, the values of p, q, and r must all be less than 4 and for $P(D \text{ beat } A) = 2/3$ those values must all be greater than 2. Thus the only integer value for all three is 3.

16. **56** The first three cuts results in 2 pieces, then 4 and then 7 pieces as can be seen in the diagram. Since each new slice should intersect each of the existing slice lines, the number of pieces will grow by 4, hence 11 pieces. This pattern continues producing the sequence 2, 4, 7, 11, 16, 22, 29, 37, 46, 56. An alternative to this recursive process is to use the formula for the sum of an arithmetic sequence, $S = \frac{n}{2}(a + l)$, where n is the number of terms (9), a is the first term (2) and l is the last term (10). This sum must be added to the initial value of 2 to get 56.



17. **35** Since $(a+3)^2 + (b+3)^2 = 289 \rightarrow a^2 + 6a + 9 + b^2 + 6b + 9 = 289 \rightarrow a^2 + b^2 + 6(a+b) = 271$. We also know that $a^2 + b^2 = 169$, so $6(a+b) = 102 \rightarrow a + b = 17$. We wish to determine the value of $\sqrt{(a+16)^2 + (b+16)^2} = \sqrt{a^2 + 32a + 256 + b^2 + 32b + 256} = \sqrt{a^2 + b^2 + 32(a+b) + 512}$. Using substitution we get $\sqrt{169 + 32(17) + 512} = \sqrt{1225} = 35$.

18. **54** The bisector of an angle of a triangle divides the opposite side of the triangle into 2 parts in the same ratio as the corresponding adjacent sides of the triangle. Therefore $\frac{AD}{DB} = \frac{AC}{BC}$. If $AD = x$ then $BD = 52 - x$. Solving the proportion for x we find that $x = 16$. Since the perimeter of $\Delta BCD = 99$. $CD = 18$ and the perimeter of $\Delta ADC = 20 + 16 + 18 = 54$.

19. **5** The only way that three absolute expressions could have a sum of zero is if each expression was zero. Thus $3y+4x-4=0$, $|x|-x=0$, and $|x-3|+x-3=0$. The first equation is a line, the second results in $x \geq 0$, and the third results in $x \leq 3$. To satisfy all three gives us a line segment whose endpoints are (0, 4/3) and (3, –8/3). Apply the distance $\sqrt{3^2+(-4)^2}=5$.

20. **321** Let the first equation be A the second B and the third C. Then find $A - 3B + 3C$. This results in $10x_1+15x_2+21x_3+28x_4+36x_5=321$. The formula $A - 3B + 3C$ can be found by inspection (notice the pattern in the coefficients of the equations) or by solving the system of equations: $a+3b+6c=10$, $3a+6b+10c=15$, and $6a+10b+15c=21$.

Tie Breakers - NMT 2011 **Solutions**

1.

2.

3.