Nassau County Interscholastic Mathematics League

9 | **Grade 9** | **TEAM #** ☐

Mathematics Tournament 2007

No calculators may be used on this part.
All answers will be integers from 0 to 999.
One (1) point for each correct answer.

Name ____________________ **School** ______________ **Score** ________

Time Limit: 45 minutes **Lower Division** *Answer Column*

1. Compute the sum of the roots of the equation $4x^2 - 8x + 3 = 0$.	1.
2. The mean of { 8, 15, x, 44, 56 ,61} is 34. Compute the median.	2.
3. The area of a circle is 144π. Compute the quotient when the area of the circle is divided by its circumference.	3.
4. If $xyz = 0$ and $wxy = 4$, compute $z(w+x+y)$.	4.
5. What is the total number of integral values that satisfy the inequality $20 \le x^2 \le 121$?	5.
6. If $3.6x^2y^7k = 21.6x^5y^9$ for all values of x and y, where k is expressed as cx^ay^b, compute $a + b + c$.	6.
7. If $18x + 27y = 81$, compute $24x + 36y$.	7.
8. A right triangle is inscribed in a circle whose radius is 5. One of the legs of the triangle is 6. If the area of the region outside the triangle but inside the circle is computed, the result can be written in the form $a\pi - b$, where a and b are integers. Compute $a + b$.	8.
9. Alice's age is 16 more than the sum of Barbara's and Cathy's ages. The square of Alice's age is 1632 more than the square of the sum of Barbara's and Cathy's ages. Calculate the sum of the ages of the three girls.	9.
10. The endpoints of a segment on the real number line are $\frac{4}{7}$ and $\frac{4}{3}$. The segment is divided into 4 equal parts. If $\frac{a}{b}$ represents the coordinate of the point closer to the larger endpoint, in simplest form, find a.	10.

Mathematics Tournament 2007

9 | **Grade 9**

Time Limit: 45 minutes **Lower Division** *Answer Column*

11. If (x # y) is defined as $\frac{4x + y + 1}{2x + y - 3}$, compute ($\sqrt{3}$ # 7) in simplest form.	11.
12. The number of dog owners in New City increased by 100 and then decreased by 15%. The city now has 56 fewer dog owners than it did before the increase by 100. Compute the original number of dog owners in New City?	12.
13. The length of each side of a square-shaped pond is 12m. The height of a reed growing from the center of the pond is 2 meters higher than the pond's surface. (The reed grows from the bottom of the pond). When the top of the reed is pulled to a corner of the pond, the reed just touches the surface. The depth of the pond is x meters. Compute x.	13.
14. Points B, C, D, and E lie on a line, in that order, with $BC = DE$ and $CD = 12$. Point A is not on the line and $AC = AD = 10$. The perimeter of ΔABE is twice the perimeter of ΔACD. Compute BC.	14.
15. In trapezoid $TRAP$ with bases TR and AP, $TR = 52$, $RA = 12$, $AP = 39$, and $PT = 5$. Compute the area of $TRAP$.	15.

Nassau County Interscholastic Mathematics League

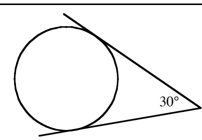
10 | **Grade 10** | **TEAM #** ☐

Mathematics Tournament 2007

No calculators may be used on this part.
All answers will be integers from 0 to 999.
One (1) point for each correct answer.

Name ____________________ **School** _______________ **Score** _________

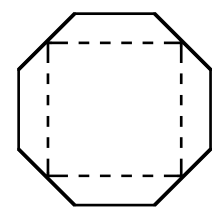
Time Limit: 45 minutes **Lower Division** *Answer Column*

1. Find the number of cubic meters contained within a rectangular prism whose edges measure 400 centimeters, .05 kilometers, and 2 meters.	1.
2. Let A be a positive acute angle such that $\sin A = \cos A$. Find the degree measure of $\angle A$.	2.
3. Ryan calculates the volume of a cube to be x cubic inches. Allison calculates the surface area of the same cube to be x square inches. If they are both correct, find the value of x.	3.
4. Two tangents drawn to a circle form an angle of 30º. Compute the positive degree difference between the major and minor arcs. 	4.
5. Two similar polygons are drawn. The area of the second polygon is 21% more than the area of the first, while the perimeter of the second polygon is x% more than the first. Compute x.	5.
6. What is the total number of integers that satisfy the equation $\sqrt{x^2 - 6x} < 4$?	6.
7. Point P with coordinates (6, 0) is rotated 45° counterclockwise about the origin. If the coordinates of P' are (x, y), compute y^2.	7.
8. A clearance item at an electronics store shows an original price of \$250. The final sale price of \$108 was the result of 3 consecutive markdowns of 10%, followed by 20%, followed by x%. Solve for x.	8.
9. Let a/b be the simplified fraction whose decimal representation is $.8\overline{33}$. Compute $b^2 - a^2$.	9.

Mathematics Tournament 2007

10 | **Grade 10**

Time Limit: 45 minutes **Lower Division** *Answer Column*

10. Point P is to be placed on a coordinate plane such that: 1. P must be 1 unit away from the circle $x^2 + y^2 = 16$. 2. P must be 1 unit away from the line $y = 2$. How many possible locations are there for point P?	10.
11. On a coordinate plane, two opposite vertices of a square are (–1, 0) and (7, –4). The vertex of the square that lies in the first quadrant has coordinates (x, y). Compute the product xy.	11.
12. Determine the units digit of 2007^{2007}.	12.
13. The median of 30 consecutive odd integers is 8. Compute the positive difference between the upper quartile and the lower quartile.	13.
14. A local high school allows its basketball and football players to pick the numbers to be sewn onto their jerseys. Basketball numbers must be two-digit numbers, all digits must be 0, 1, 2, 3, 4, or 5, and repetition of digits is allowed. Football numbers must be two-digit numbers, any digit 0 – 9 may be used, but repetition is not allowed. In either case 0 may appear in either the tens or units position. The number of possible basketball jerseys is what **percent** of the number of possible football jerseys?	14.
15. A banquet hall is in the shape of a regular octagon whose sides measure 20 ft. The dance floor is a square formed by joining the midpoints of every other side of the octagon. The dining area is comprised of the 4 isosceles trapezoids that are outside the square. By how many square feet does the dance floor's area exceed the total dining area? 	15.

Nassau County Interscholastic Mathematics League

11

Grade 11

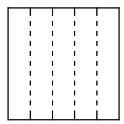
TEAM #

Mathematics Tournament 2007

No calculators may be used on this part.
All answers will be integers from 0 to 999.
One (1) point for each correct answer.

Name ______________________ **School** _______________ **Score** _________

Time Limit: 45 minutes **Upper Division** *Answer Column*

1. The number $\sqrt{112}-\sqrt{63}+\sqrt{343}$ can be simplified to a number in the form $a\sqrt{b}$. Compute $a + b$.	1.
2. If $a\otimes b=\frac{a+b}{b-a}$, compute the value of $(3\otimes 5)\otimes 12$.	2.
3. A square is cut into five rectangles using lines parallel to one of the sides of the square. The area of each resulting rectangle is 80. Compute the perimeter of the original square. 	3.
4. A triangle is formed in quadrant II by the coordinate axes and the line $ax+by=-12$. If the area of the triangle is 12 and $\|a\|=\|6b\|$, compute the value of $a - b$.	4.
5. The $\sqrt{(7+24i)}$ can be written in the form $a+bi$, where $i=\sqrt{-1}$. Compute ab.	5.
6. How many *lattice points* are on the line segment whose endpoints are (8, 22) and (53, 286)? A *lattice point* is a point on the coordinate plane that has integer values.	6.
7. If $3\sin x=4\cos x$, compute $100\sin x\cos x$.	7.
8. An experiment is performed with only four possible outcomes none of which can occur simultaneously. The probability that outcome A occurs is $1/10$, the probability B occurs is $1/4$, and the probability that C does not occur is $5/12$. Compute the reciprocal of the probability that D does occur.	8.

Mathematics Tournament 2007

11

Grade 11

Time Limit: 45 minutes **Upper Division** *Answer Column*

9. The graph of the equation $2x-5y+12=0$ is translated 4 units to the right and the resulting graph is then reflected over the *x*-axis. The final graph has an equation in the form $ax+by+c=0$, where *a, b,* and *c* are relatively prime (GCF = 1). Compute $a + b + c$.	9.
10. Given $\cot 260°+\tan 130°=\csc x°$, compute the least positive value of *x*.	10.
11. Solve for *x*: $\log_{10}\left(x^3\right)=\frac{6}{\log_x(5x)}$.	11.
12. The area of parallelogram *ABCD* is 56 and $AB=8$ and $BC=10$. Point *E* is on side $\overline{AB}$, point *F* is on side $\overline{BC}$, and point *G* is on side $\overline{AD}$ such that $AE=BF=AG=4$. Given that the line through *G* parallel to $\overleftrightarrow{EF}$ intersects $\overline{CD}$ at point *H*, compute the area of quadrilateral *EFHG*.	12.
13. The arithmetic mean of 13 different integer scores is 80 and the median is 85. If the range of the scores is defined as the absolute value of the difference between the largest and smallest scores, compute the <u>minimum</u> range.	13.
14. Triangle ABC is inscribed in a circle whose radius is 10. The three arcs formed by the vertices of the triangle have lengths in the ratio of 3:4:5. Find the sum of the squares of the lengths of the two shortest sides of the triangle.	14.
15. Compute the sum of the roots of $\tan^2 x=4\tan x-1$ given that $0°\le x\le 360°$.	15.

Nassau County Interscholastic Mathematics League

12 | **Grade 12** | **TEAM #** ☐

Mathematics Tournament 2007

No calculators may be used on this part.
All answers will be integers from 0 to 999.
One (1) point for each correct answer

Name ____________________ **School** ______________ **Score** ________

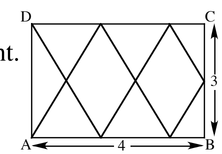
Time Limit: 45 minutes **Upper Division** *Answer Column*

1. Let $p(x) = x^2 - 16x + 61$. Compute the product of all the values of x for which $p(x) = p'(x)$.	1.
2. Given $\begin{cases} a+b+c-d=74 \\ a+b-c+d=86 \\ a-b+c+d=94 \\ -a+b+c+d=106 \end{cases}$, compute the value $a+b+c+d$.	2.
3. Compute $19\lim_{x\to 0}(21x\csc x)$.	3.
4. Compute the sum of the infinite geometric series $81+54+36+24+\ldots$.	4.
5. If $f(x) = \lvert 6x - x^2 - 9\rvert$, compute $f'(4)$.	5.
6. Let G be the graph of the polar curve $r = \theta^3 - 6\theta^2 + 9\theta$. Compute the number of times that G passes through the origin.	6.
7. The symbol $f^{(n)}(x)$ indicates the n^{th} derivative of the function f. Let $f(x) = e^{9x}$. Compute $\frac{1}{6}\log_3\left(\frac{f^{(2007)}(x)}{f(x)}\right)$.	7.
8. Compute the number of integers in the domain of $f(n) = \dfrac{\tan\left(\frac{\pi n}{2}\right)}{\sqrt{400-n^2}}$.	8.

Mathematics Tournament 2007

12 | **Grade 12**

Time Limit: 45 minutes **Upper Division** *Answer Column*

9. A line is drawn tangent to the graph of $y = \frac{1}{x}$ at some point on the curve in the first quadrant. This line forms a triangle with the x- and y-axes. Let A be the area of this triangle. Compute $100A$, rounded to the nearest integer.	9.
10. The circles $x^2 + y^2 = 36$ and $x^2 + y^2 - 10x - 24y + k = 0$ are externally tangent. Compute k.	10.
11. A region is bounded by the graphs of $y = x^2$, $y = x^2 + 5$, $x = -2$, and $x = 2$. Compute the area of this region.	11.
12. A scientist fires a laser from point A inside a mirrored rectangular box, as shown at the right. If $AB = 4$, $BC = 3$, and if the beam reflects as illustrated, compute the total distance that the beam travels before reaching point D. 	12.
13. Let b be a number chosen at random from the set {2, 3, 4, …, 2007}. Let P be the probability that the average rate of change of $f(x) = x^5$ over the interval $1 \le x \le b$ is an integer. Compute $118P$.	13.
14. A sequence $\{a_n\}_{n=0}^{\infty}$ is formed such that $a_0 = 0$, $a_1 = 1$, and, for all integers $n \ge 0$, the subsequence $\{a_n, a_{n+1}, a_{n+2}\}$ forms: * an arithmetic sequence if n is even * a geometric sequence if n is odd. Compute a_{61}.	14.
15. A line, normal to the graph of $y = x^4$ at a point other than the origin, will cross the y-axis at a single point (0, b). The minimum possible value of b can be written as a fraction p/q in simplest terms, with $q > 0$. Compute $p + q$.	15.

Nassau County Interscholastic Mathematics League

M | **Mathletics** | **TEAM #** ____

Mathematics Tournament 2007

Calculators may be used on this part.
All answers will be integers from 0 to 999.
One (1) point for each correct answer.

Name ______________ **School** ______________ **Score** ________

Time Limit: 30 minutes | **Both Divisions** | *Answer Column*

1. Two rectangles have the same area. If the base of the first rectangle is 25% larger than the base of the second, then the height of the first rectangle is k% smaller than the height of the second. Compute k.	1.
2. For how many distinct integers, d, is $\frac{2007}{d}$ an integer?	2.
3. Given $x+y-z=25$ and $x+y+2z=40$, compute the value of $x+y$.	3.
4. ΔNMT has vertices $N(0,0)$, $M(5,5)$, and $T(15,-5)$. $\Delta N'M'T'$ is the image of ΔNMT after the transformation $\left(R_{90^\circ} \circ D_2 \circ r_{y=x}\right)$ where the center of the rotation and the dilation is the origin. Compute the number of square units in the area of $\Delta N'M'T'$.	4.
5. Evaluate $\left(8^{-1}+4^{-2}+(-2)^{-3}\right)^{-2}$.	5.

Mathematics Tournament 2007

M | **Mathletics**

Time Limit: 30 minutes | **Both Divisions** | *Answer Column*

6. Given the sum of two numbers is 3 and their product is –12, compute the sum of their cubes.	6.
7. Sunny and Matt compete in a math contest that has 25 questions. All questions are equally difficult. Sunny answered 20 correctly and Matt answered 15 correctly. What is the most likely number of questions that were correctly answered by both students?	7.
8. When the expression $(x+2y+3z)^{10}$ is expanded one of the terms can be expressed as $n \cdot x^8 yz$. Compute the value of n.	8.
9. Two parallel chords are drawn 6 inches apart in a circle whose diameter is 12. Let k be the area of the portion of the circle that lies between the two chords. The maximum value of k can be written in the form $a\pi + b\sqrt{p}$, where a, and b are integers and p is prime. Compute $a+b+p$.	9.
10. The expression $\sqrt{1+2\sqrt{1+2\sqrt{1+2\sqrt{1+\ldots}}}}$ can be simplified into $a+\sqrt{b}$. Compute $a+b$.	10.

Nassau County Interscholastic Mathematics League

T

Team Problem Solving

TEAM #

Mathematics Tournament 2007

HAND IN ONLY **ONE** ANSWER SHEET PER TEAM
Calculators may be used on this part.
All answers will be integers from 0 to 999.
Three (3) points per correct answer.

Name ____________ **School** ____________ **Score** ________

Time Limit: 60 minutes **Both Divisions** *Answer Column*

1. What three-digit number is eleven times the sum of its digits?	1.
2. Compute the sum of the squares of two numbers whose sum is 10 and whose product is 32.	2.
3. The perimeter of a right triangle is 36. The sum of the squares of its sides is 450. Determine the area of the triangle.	3.
4. The bases of an isosceles trapezoid measure 6 and 24. Compute the exact quotient of the number of square units in the area of its inscribed circle divided by the number of units in the circumference of its inscribed circle.	4.
5. For n, a positive integer, the expression $1^n + 2^n + 3^n + 4^n$ is divisible by 5 for all n except when n is divisible by an integer k. Determine k.	5.
6. Compute the absolute value of the difference between the smallest positive value of n and the largest negative value of n for which $n^2 - 8n + 89$ is divisible by 73.	6.
7. Determine the average of the values of x for which $\left(x^2 - 9x + 19\right)^{x^2 - 6x - 91} = 1$.	7.
8. Let S = the total of all the digits in the first 9,999 positive integers. Compute $\frac{S}{1000}$.	8.
9. Compute the value of the infinitely long nested radical $\sqrt{56 + \sqrt{56 + \sqrt{56 + \sqrt{56 + \ldots}}}}$.	9.
10. If $2^{2x+3} = 14$, compute the exact numerical value of $10\left(2^{4x+3}\right)$.	10.

T

Team Problem Solving

Time Limit: 60 minutes **Both Divisions** *Answer Column*

11. In a four-player game, the odds against Amy winning are 4:5, the odds against Beth winning are 5:2, the odds against Calvin winning are 8:1, and the odds against Debbie winning are x:1. If exactly one of these players will win the game, compute x	11.
12. There are three real values of x, one positive and two negative, for which $x^3 + 11x^2 - 25x = 275$. Compute the product of the two negative values.	12.
13. Determine the exact area of the region enclosed by the graph of $\left\|\frac{x}{2}\right\| + \left\|\frac{y}{2}\right\| = 9$.	13.
14. A diagonal of a rectangular prism is 11 units long. If two of its dimensions are $2\sqrt{17}$ and $2\sqrt{7}$, how many units long is the third dimension?	14.
15. The volume of a cube is $\frac{125}{27}\,yd^3$. Determine the number of square feet in the surface area of the cube.	15.
16. Kevin and Tom are standing next to each other at the edge of a lake that is exactly one-half mile around. At 2 PM, Tom begins to walk at a pace of 4 mph and Kevin begins to walk at 3.5 mph in opposite directions around the lake. How many times will Tom and Kevin meet between 2:01 PM and 2:59 PM?	16.
17. Given: $\begin{cases} A + 2B + 2C = 25 \\ 2A + 3B + D = 29 \\ A + C + 2D = 11 \end{cases}$. Compute $A + B + C + D$.	17.
18. A certain U.S. city has experienced a population growth of 5% for each of the last five years. If, at the beginning of 2007, its population was 1,664,775, and its population at the beginning of 2005 was represented as a number in scientific notation, in the form $a \times 10^n$, compute $100 \cdot a \cdot n$.	18.
19. What is the total number of integers for which $\left\|x^2 - 4x - 165\right\| \neq x^2 - 4x - 165$?	19.
20. What is the remainder when 2^{23} is divided by 511?	20.

1. **2** For quadratic equations the sum of the roots is $\frac{-b}{a}$.

2. **32** Determine x using $x + 184 = 6(34)$. So $x = 20$. The median of 6 scores is the mean of the two middlemost scores, 20 and 44.

3. **6** Since the area of the circle is 144π, the radius is 12 and the circumference is 24π.

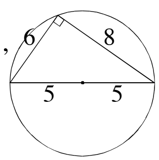
4. **0** Since $wxy = 4$, $xy \neq 0$, but $xyz = 0$, so $z = 0$ and thus $z(w+x+y) = 0$.

5. **14** Solve for x: $\sqrt{20} \le x \le 11$ or $-11 \le x \le -\sqrt{20}$. Since $\sqrt{20} \approx 4.5$, $x \in \pm\{5,6,7,8,9,10,11\}$.

6. **11** $k = \frac{21.6x^5y^9}{3.6x^2y^7} = 6x^3y^2$, therefore $a + b + c = 3 + 2 + 6 = 11$.

7. **108** Multiply the original equation by $\frac{4}{3}$: $24x + 36y = 108$.

8. **49** The $A_{circle} = 25\pi \approx 78.54$ and $A_{\triangle ABC} = \frac{1}{2}(6)(8) = 24$, so the difference is $25\pi - 25$ and $a + b = 49$.



9. **102** $a = b + c + 16 \Rightarrow a - (b + c) = 16$ and $a^2 = (b+c)^2 + 1632 \Rightarrow a^2 - (b+c)^2 = 1632$. Factoring the last equation results in $(a + (b+c))(a - (b+c)) = 1632$ and substituting we find $16(a + b + c) = 1632$. Therefore $a + b + c = 102$.

10. **8** Method 1: The mean of $\frac{4}{7}$ and $\frac{4}{3}$ is $\frac{20}{21}$ and the mean of $\frac{20}{21}$ and $\frac{4}{3}$ is $\frac{24}{21} = \frac{8}{7}$.

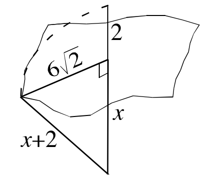
Method 2: Since $\frac{1}{4}\left(\frac{4}{3} - \frac{4}{7}\right) = \frac{4}{21}$, and $\frac{4}{3} - \frac{4}{21} = \frac{24}{21} = \frac{8}{7}$ $a = 8$.

Grade 9 - NMT 2005 Solutions

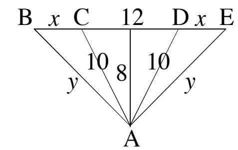
11. **2** $\frac{4\sqrt{3} + 7 + 1}{2\sqrt{3} + 7 - 3} = \frac{4\sqrt{3} + 8}{2\sqrt{3} + 4} = 2$.

12. **940** Let x = no. of dog owners. Then $.85(x + 100) = x - 56$. Solve for x: $.15x = 141$, so $x = 940$.

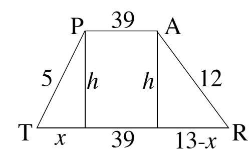
13. **17** The diagonal length of the pond is $12\sqrt{2}$. Apply the Pythagorean theorem: $x^2 + \left(6\sqrt{2}\right)^2 = (x+2)^2$ and solve for x. $x^2 + 72 = x^2 + 4x + 4 \Rightarrow 4x = 68 \Rightarrow x = 17$.



14. **9** $P_{\triangle ABE} = 2x + 2y + 12$ and $P_{\triangle ACD} = 32$. It follows that $P_{\triangle ABE} = 2P_{\triangle ACD} = 64$ and $x + y = 26$. By the Pythagorean theorem or by recognizing the Pythagorean triple 6-8-10, the distance from A to $\overline{CD}$ is 8. Using the Pythagorean theorem again we find $8^2 + (6+x)^2 = (26-x)^2 \Rightarrow x = 9$.



15. **210** $A_{trap} = \frac{h}{2}(b_1 + b_2) = \frac{h}{2}(39 + 52) = \frac{91h}{2}$. By Pythagorean theorem: $h^2 + x^2 = 5^2$ and $h^2 + (13-x)^2 = 12^2$. Solving for x we get $x = \frac{25}{13}$, and then solving for h we find $h = \frac{60}{13}$ and therefore A = 210.



1. **400** Convert 400 cm = 4 m, .05 km = 50 m, then (4)(50)(2) = 400.

2. **45** If sin *A* = cos *A* then, in a right triangle, the adjacent side must equal the opposite side which means the two acute angles are each 45°.

3. **216** Let *a* be an edge of the cube. Then $a^3 = 6a^2 \Rightarrow a = 6 \Rightarrow a^3 = 6^3 = 216$.

4. **60** The given angle is measured by one-half the difference of the two intercepted arcs. Thus $30 = \frac{1}{2}(x - y) \Rightarrow x - y = 60$.

5. **10** The ratio of the areas of two similar polygons equals the ratio of the squares of their respective perimeters. Therefore $\frac{A_1}{A_2} = \frac{P_1^2}{P_2^2} \Rightarrow \frac{1}{1.21} = \frac{1^2}{(1+x)^2} \Rightarrow (1+x)^2 = 1.21 \Rightarrow x = .1 = 10\%$

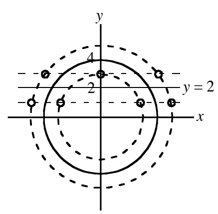
6. **4** Since $\sqrt{x^2 - 6x} < 4 \Rightarrow 0 \le x^2 - 6x < 16$. Solving each inequality separately we find that $0 \le x^2 - 6x$ for $x \le 0$ or $x > 6$, and $x^2 - 6x < 16 \Rightarrow x^2 - 6x - 16 < 0$ when $-2 < x < 8$. There are 4 integer values in this solution set {–1, 0, 6, 7}.

7. **18** The coordinates of P' are $(3\sqrt{2}, 3\sqrt{2})$ and $(3\sqrt{2})^2 = 18$.

8. **40** Solve: $250(.90)(.80)(100 - x) = 108 \Rightarrow$ $180(100 - x) = 108 \Rightarrow 100 - x = 60 \Rightarrow x = 40$

9. **11** Convert $.8\overline{33}$ to a fraction as follows: Let $n = .8\overline{33}$ then $10n = 8.\overline{33} \Rightarrow 10n - n = 8.\overline{33} - .8\overline{33} = 7.5$ so $n = \frac{7.5}{9} = \frac{15}{18} = \frac{5}{6}$ and $5 + 6 = 11$.

10. **7** The set of points (locus) that satisfy the first condition is two circles concentric to the given circle and 1 unit from it. The set of points satisfying the second condition are 2 lines parallel to the given line and each 1 unit from it. These circles and lines have 7 points in common as seen in the accompanying diagram.



11. **10** In a square, the diagonals are perpendicular bisectors of each other. Therefore the midpoint of the segment joining the two given points, (3, –2), is also the midpoint of the second diagonal. In addition the slopes of the diagonals are negative reciprocals. From (3, –2) to (7, –4) we get $(\Delta x, \Delta y) = (4, -2)$ so to calculate the point in quadrant I we set $(\Delta x, \Delta y) = (2, 4)$. Thus the point is (5, 2) and the product of the coordinates is 10.

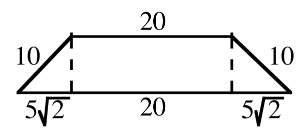
12. **3** Raising 7 to consecutive integer powers results in the following pattern for the units digits: 7, 9, 3, 1, 7, 9, … . This cycle of 4 values means that when 2007 is raised to the 2004 power the units digit is a 1. After 3 more multiplications it becomes a 3.

13. **30** Since 8 is the median of 30 consecutive odd integers, there are 15 integers on each side of 8. They range from –21 to 7 and from 9 to 37. The lower quartile is the median of the first set and the upper quartile is the median of the second set of integers. 23 – (–7) = 30.

14. **40** The number of basketball jerseys is $6 \times 6 = 36$ and the number of football jerseys is $10 \times 9 = 90$. To determine what percent of 90 is 36 use $\frac{36}{90} = \frac{2}{5} = 40\%$.

15. 400 Method 1: Examine the top trapezoidal shape. Since the height of the trapezoid is $5\sqrt{2}$ we can calculate

$A_{trap} = \frac{5\sqrt{2}}{2}(40 + 10\sqrt{2}) = 100\sqrt{2} + 50$



and

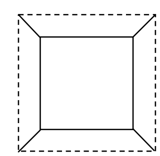
$A_{square} = (20 + 10\sqrt{2})^2 = 400 + 400\sqrt{2} + 200$

. The difference in the areas is

$A_{square} - 4A_{trap} = (600 + 400\sqrt{2}) - 4(100\sqrt{2} + 50) = 600 - 200 = 400$

.

Method 2: Reflect each of the trapezoids along their longer bases to obtain the diagram at right. The area of the small square at right is equal to the area of the dance floor (the large square) minus the area of the dining area (the 4 trapezoids,) which is the exact quantity asked for. Since the sides of the small square are also sides of the octagon, they each have length 20. Therefore, the answer is (20)(20) = 400.



1. **15** Simplify as follows: $4\sqrt{7}-3\sqrt{7}+7\sqrt{7}=8\sqrt{7}$. Therefore a = 8 and b = 7.

2. **2** By substitution: $(3\otimes 5)\otimes 12=\left(\dfrac{\left(\dfrac{3+5}{5-3}\right)+12}{12-\left(\dfrac{3+5}{5-3}\right)}\right)=\dfrac{16}{8}=2$

3. **80** Let the short side of each rectangle be x then $5x$ is the length of each longer side. Solve $5x\cdot x=80$ so x = 4 and the perimeter of the original square is 4(20) = 80.

4. **7** The x- and y-intercepts are $\dfrac{-12}{a}$ and $\dfrac{-12}{b}$ respectively. Then $A=\dfrac{1}{2}\left(\dfrac{-12}{a}\right)\left(\dfrac{-12}{b}\right)=\dfrac{72}{ab}=12$. Therefore ab = 6. Since $|a|=|6b|$ it follows that b = –1 and a = 6, so $a-b$ = 7.

5. **12** Let $a+bi=\sqrt{7+24i}$. Square both sides to get $a^2-b^2+2ab=7+24i$. By the equality of complex numbers, $a^2-b^2=7$ and $2ab=24$. Thus ab = 12.

6. **4** The equation of the line containing the two given points is $y-22=\dfrac{88}{15}(x-8)$. Thus $(x-8)$ must be divisible by 15, with $0\le x\le 53$. The only such values are 8, 23, 38, and 53.

7. **48** $3\sin x=4\cos x\Rightarrow \tan x=\dfrac{4}{3}$. Using SOHCAHTOA, we find $\sin x=\dfrac{4}{5}$ and $\cos x=\dfrac{3}{5}$ so $100\sin x\cos x=100\left(\dfrac{4}{5}\right)\left(\dfrac{3}{5}\right)=48$

8. **15** $P(A)+P(B)+P(C)+P(D)=1$, so $P(D)=1-\left(\dfrac{1}{10}+\dfrac{1}{4}+\dfrac{7}{12}\right)=1-\dfrac{56}{60}=\dfrac{4}{60}$.

9. **11** To translate a relation 4 units to the right replace x with $x-4$ and to reflect it over the x-axis, replace y with $-y$. This results in the equation $2(x-4)-5(-y)+12=0\Rightarrow 2x+5y+4=0$.

Grade 11 - NMT 2005 Solutions

10. **260**

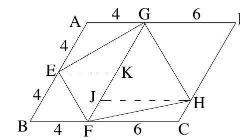
$$\cot 260^\circ+\tan 130^\circ=\frac{\cos 260^\circ}{\sin 260^\circ}+\frac{\sin 130^\circ}{\cos 130^\circ}=\frac{\cos 260^\circ\cos 130^\circ+\sin 130^\circ\sin 260^\circ}{\sin 260^\circ\cos 130^\circ}$$

. Applying the cos(A + B) formula results in

$$\cot 260^\circ+\tan 130^\circ=\frac{\cos(260^\circ-130^\circ)}{\sin 260^\circ\cos 130^\circ}=\frac{1}{\sin 260^\circ}=\csc 260^\circ.$$

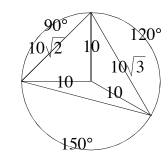
11. **20** $\log_{10}(x^3)\cdot\log_x(5x)=\dfrac{3\log x}{\log 10}\cdot\dfrac{\log(5x)}{\log x}=3\cdot\log(5x)=6$. Thus $5x=10^2=100$ and x = 20.

12. **28** Through points E and H construct lines parallel to $\overline{BC}$. This results in four parallelograms demonstrating that the area of quadrilateral *EFGH* is one-half the area of parallelogram *ABCD*.



13. **23** Place the numbers in size order: $x_1, x_2, x_3, \ldots, x_{13}$. Then $x_1+x_2+x_3+\ldots+x_{13}=80(13)=1040$ and $x_7=85$. In order to minimize the range minimize the top half of the numbers. They form the set {86, 87, 88, 89, 90, 91}. It is also necessary to make x_1 as great as possible to minimize the difference between the largest and smallest scores. If we select the set {68, 69, 70, 71, 72, 74} we satisfy the mean and have the minimum range. 91 – 68 = 23

14. **500** The arcs and their corresponding central angles measure 90°, 120°, and 150°. Therefore the sides opposite the two smaller angles are of length $10\sqrt{2}$ and $10\sqrt{3}$. The sum of their squares is $(10\sqrt{2})^2+(10\sqrt{3})^2=200+300=500$.



15. **540** Using the identity $\tan^2 x+1=\sec^2 x$ and placing the result in terms of sin and cos, we get $\sec^2 x=4\tan x\Rightarrow \dfrac{1}{\cos^2 x}=\dfrac{4\sin x}{\cos x}$. Since $\cos x\ne 0$, this becomes $4\sin x\cos x=1$ which simplifies to $2\sin x\cos x=\dfrac{1}{2}\Rightarrow \sin(2x)=\dfrac{1}{2}$ and so $2x$ = 30°, 150°, 390°, 510°…. Based upon the domain, we have x = 15°, 75°, 195°, and 255°.

1. **77** $p(x) = p'(x) \Rightarrow x^2 - 16x + 61 = 2x - 16 \Rightarrow x^2 - 18x + 77 = 0$. The product of the roots = 77.

2. **180** Add all four equations to get:
$2a + 2b + 2c + 2d = 360 \Rightarrow a + b + c + d = 180$.

3. **399** $19 \lim_{x \to 0} (21x \csc x) = 19 \cdot 21 \lim_{x \to 0} \left(\frac{x}{\sin x} \right) = 399(1) = 399$

4. **243** The sum of an infinite geometric series:
$S = \frac{a}{1-r} = \frac{81}{1 - 2/3} = \frac{81}{1/3} = 243$.

5. **2** <u>Method 1</u>: The absolute value negates the slope when the function has negative outputs. Since this is the case for all $x \neq 3$,
$f'(x) = \begin{cases} 6 - 2x, x \leq 3 \\ 2x - 6, x > 3 \end{cases}$. Therefore, $f'(4) = 2$.
<u>Method 2</u>:
$f(x) = |6x - x^2 - 9| = |-(x^2 - 6x + 9)| = |-(x-3)^2| = (x-3)^2$. So
$f'(x) = 2(x-3)$.

6. **2** A polar curve passes through the origin when $r = 0$. Therefore factor to solve: $\theta^3 - 6\theta^2 + 9\theta = 0$.

7. **669** Examine the pattern of derivatives:
$f'(x) = 9e^{9x}, f''(x) = 9^2 e^{9x}, \ldots, f^{(n)}(x) = 9^n e^{9x}$. So
$$\frac{1}{6}\log_3\left(\frac{f^{(2007)}(x)}{f(x)}\right) = \frac{1}{6}\log_3\left(\frac{9^{2007}e^{9x}}{e^{9x}}\right) = \frac{1}{6}\log_3\left(9^{2007}\right) = \frac{2007}{6}\log_3 9$$
$$= \frac{669}{2}(2) = 669.$$

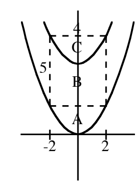
8. **19** The tan function is undefined at $\theta = \frac{n\pi}{2}$, where n is an odd integer.
$400 - n^2 > 0$, $-20 < n < 20$ and $n \in \{0, \pm 2, \pm 4, \pm 6, \ldots, \pm 18\}$.

Grade 12 - NMT 2005 **Solutions**

9. **200** The equation of the tangent line at the point $\left(a, 1/a\right)$ can be found as follows: $f'(a) = \frac{-1}{a^2}$, so $y - 1/a = \frac{-1}{a^2}(x - a)$ is the equation.
Calculating the x- and y-intercepts we get $(2a, 0)$ and $\left(0, 2/a\right)$. Thus $A = \frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$. [Note: since the answer is numerical, any point such as (1, 1) could have been used.]

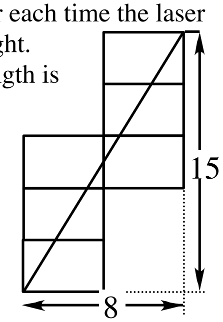
10. **120** Complete the square for the second circle:
$x^2 - 10x + 25 + y^2 - 24y + 144 = -k + 25 + 144$ or
$(x-5)^2 + (y-12)^2 = 169 - k$. Therefore the centers of the circles are at (0, 0) and (5, 12). The distance between these points = 13. Since the radius of the first circle is 6, the radius of the second circle must be 7, so $169 - k = 49$ and $k = 120$.

11. **20** <u>Method 1</u>: Consider the diagram at the right. The parabolas are congruent and 5 units apart. Area A is identical to area C. Thus $A + B = B + C$. The area is (4)(5) = 20



<u>Method 2</u>: $\int_{-2}^{2} \left(x^2 + 5 - x^2\right) dx = \int_{-2}^{2} 5\,dx = 5x\Big|_{-2}^{2} = 20$.

12. **17** <u>Method 1</u>: Reflect the rectangle along a mirror each time the laser hits la mirror, to produce the diagram at the right.
By the Pythagorean theorem, the total path length is 17.
<u>Method 2</u>: Each portion of the path is a hypotenuse of a triangle with legs $4/5$ and $3/2$. There are 10 triangles having hypotenuse = $17/10$.



13. **118** Average rate of change =
$\frac{f(b) - f(1)}{b - 1} = \frac{b^5 - 1}{b - 1} = b^4 + b^3 + b^2 + b + 1$. Since b is an integer the result is integral. So $P = 1$ and $118P = 118$.

14. **961** Apply the rules to create a table of values as seen below. Notice that when n is odd, a_n is a perfect square, that is $a_{2n-1} = n^2$. So $a_{61} = a_{2(31)-1} = 31^2 = 961$.

n	0	1	2	3	4	5	6	7	8	9
a_n	0	1	2	4	6	9	12	16	20	25

15. **7** The slope of the normal at (a, a^4) is $\frac{-1}{f'(a)} = \frac{-1}{4a^3}$, so the equation of the normal is $y = \frac{-1}{4a^3}(x-a) + a^4 = \frac{-1}{4a^3}x + \frac{1}{4}a^{-2} + a^4$. Thus $b = \frac{1}{4}a^{-2} + a^4$. Set $\frac{db}{da} = 0$ to find the minimum. Hence, $\frac{db}{da} = \frac{-1}{2}a^{-3} + 4a^3 = 0 \Rightarrow 4a^3 = \frac{1}{2a^3} \Rightarrow a^6 = \frac{1}{8}$. Consequently $a^2 = \frac{1}{2}$, $a^{-2} = 2$, and $a^4 = \frac{1}{4}$, so min $b = \frac{1}{4}(2) + \frac{1}{4} = \frac{3}{4}$ and 3+4=7.

1. **20** Let b and h be the base and height of the second rectangle. Then the base and height of the first rectangle are (1 + .25)b and (1 – k)h. Since the areas are equal, $bh = 1.25bh(1-k\%)$. Solving for k we find that $k\% = \frac{1}{5} = 20\%$.

2. **12** Factor 2007 by recognizing it is divisible by 9, $2007 = 3^2 \cdot 223$. Form all possible divisors of 2007: ±1, ±3, ±9, ±223, ±669, ±2007.

3. **30** Multiply the first equation by 2 and add it to the second equation: $3x + 3y = 90$. Divide by 3.

4. **200** The area of ΔNMT can be found by recognizing that it is a right triangle with $\angle M$ a right angle, by using Pick's theorem, or by building a rectangle about the triangle and subtracting right triangle areas from the area of the rectangle. The area of ΔNMT = 50 square units. Of the three transformations, the only one that changes size is the dilation. Since the side lengths are doubled the area is 4 times greater or 200 square units.

5. **256** $\left(\frac{1}{8} + \frac{1}{16} - \frac{1}{8}\right)^{-2} = \left(\frac{1}{16}\right)^{-2} = 256$

6. **135** Let $a + b = 3$ and $ab = -12$. Then $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a+b)$. Using substitution we get $3^3 = a^3 + b^3 + 3(-12)(3)$ so $a^3 + b^3 = 135$.

7. **12** $P(\text{Sunny getting one right}) = \frac{20}{25} = \frac{4}{5}$ and

$P(\text{Matt getting one right}) = \frac{15}{25} = \frac{3}{5}$. Therefore

$P(\text{both getting the same right}) = \frac{4}{5} \cdot \frac{3}{5} = \frac{12}{25}$. So 12 is the most likely number correct by both.

8. **540** The multinomial expansion results in $\frac{10!}{8!1!1!}x^8(2y)^1(3z)^1 = 90(6x^8yz) = 540$.

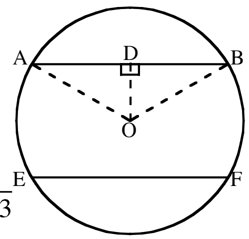
9. **33** To maximize the area between the chords they should be placed as close to the center as possible. See the diagram at the right.
$\overline{OD} \perp \overline{AB}$, $OB = 6$, and $OD = 3$. It follows that $\angle AOB = 120°$.

$A_{\sec torAOB} = \frac{1}{3}\pi(6)^2 = 12\pi$ and

$A_{\Delta AOB} = \frac{1}{2}(3)(6\sqrt{3}) = 9\sqrt{3}$. Thus

$A_{arcsegmentAB} = 12\pi - 9\sqrt{3}$ and

$A_{betweenchords} = 36\pi - 2(12\pi - 9\sqrt{3}) = 12\pi + 18\sqrt{3}$.



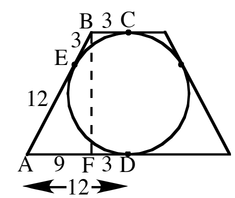
10. **3** Let $x = \sqrt{1+2\sqrt{1+2\sqrt{1+2\sqrt{1+\ldots}}}}$ then
$x = \sqrt{1+2x} \Rightarrow x^2 = 1+2x \Rightarrow x^2 - 2x - 1 = 0$. Applying the quadratic formula results in:
$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$. The negative value must be rejected leaving $x = 1 + \sqrt{2}$ and 1 + 2 = 3.

1. **198** Let h, t, and u be the hundreds', tens', and units' digits, respectively. Then $100h + 10t + u = 11(h+t+u) \Rightarrow 89h - (t + 10u) = 0$. From this it can be seen that $h = 1$, $t = 9$, and $u = 8$

2. **36** Let x and y be the two numbers so $x + y = 10$ and $xy = 32$.
Since $(x+y)^2 = x^2 + y^2 + 2xy$, by substitution,
$10^2 = x^2 + y^2 + 2(32) \Rightarrow x^2 + y^2 = 36$.

3. **54** Let the lengths of the legs be a and b and the length of the hypotenuse be c. Then $a + b + c = 39$ and $a^2 + b^2 + c^2 = 450$. Then $a^2 + b^2 = c^2 \Rightarrow 2c^2 = 450 \Rightarrow c = 15$. Since $(a+b)^2 = a^2 + b^2 + 2ab$, $(a+b)^2 = c^2 + 2ab \Rightarrow 21^2 = 15^2 + 2ab$ so $2ab = 216$ and
$A_\Delta = \frac{1}{2}ab = \frac{1}{2}(108) = 54$.

4. **3** The circle bisects the bases of the trapezoid so $BC = 3$ and $AD = 12$ as seen in the diagram. Tangents drawn from a point to a circle are equal. So $AE = 12$ and $BE = 3$.
Let $\overline{BF}$ be an altitude. Since $DF = 3 \rightarrow AF = 9$. By the Pythagorean theorem, $9^2 + BF^2 = 15^2 \Rightarrow BF = 12$.
Since BF is the length of a diameter of the circle, $r = 6$ and
$\frac{A}{C} = \frac{\pi r^2}{2\pi r} = \frac{r}{2} = 3$.



5. **4** Consider the patterns for the units digit of the integer powers of 1, 2, 3, and 4: 1 stays 1, 2 cycles through 2, 4, 6, and 8, 3 cycles 3, 9, 7, and 1, while 4 alternates between 4 and 6. Consider the sums for the n = 1, 2, 3, and 4. They all have a units digit of 0 except when n is 4. Hence when n is a multiple of 4 the sum will not be divisible by 5.

6. **146** Since $n^2 - 8n + 89 = (n-4)^2 + 73$, $n - 4$ must be divisible by 73. Therefore the largest negative value for n is –69 and the smallest positive value is 77. Their positive difference is 146.

7. **4** Let $a = x^2 - 9x + 19$ and $b = x^2 - 6x - 91$. Then $a^b = 1$ when $a = 1$, $b = 0$ and $a \neq 0$, or if $a = -1$ and b is an even integer. Solving the respective equations we find if $a = 1$, $x = 3$ or 6; when $b = 0$ then $x = -7$ or 13; and when $a = -1$, $x = 4$ or 5 (b is odd when $x = 4$ and even when $x = 5$). Hence the equation is true when for $x = -7$, 3, 5, 6, and 13. The average of these numbers is 4.

8. **180** Consider pairing up the numbers in the following way: 9999, 1 and 9998, 2 and 9997, 3 and 9996 etc. The sum of the digits in each pair is 36 and there are 5000 such pairings. It follows that:
$$\frac{S}{1000} = \frac{5000 \times 36}{1000} = 180.$$

9. **8** Let $n = \sqrt{56 + \sqrt{56 + \sqrt{56 + \sqrt{56 + \ldots}}}}$ then $n = \sqrt{56 + n}$. Solving we find $n = 8$ or -7. Reject -7.

10. **245** $2^{2x+3} = 2^{2x} \cdot 2^3 = 14 \Rightarrow 2^{2x} = \frac{14}{8} = \frac{7}{4}$.

Then $10\left(2^{4x+3}\right) = 10\left(2^{2x} \cdot 2^{2x} \cdot 2^3\right) = 10\left(\frac{7}{4}\right)^2(8) = 245$.

11. **20** $P(\text{Amy winning}) = \frac{5}{9}$, $P(\text{Beth winning}) = \frac{2}{7}$,

$P(\text{Calvin winning}) = \frac{1}{9}$, and $P(\text{Debbie winning}) = \frac{1}{x+1}$. Since

$\frac{5}{9} + \frac{2}{7} + \frac{1}{9} + \frac{1}{x+1} = 1 \Rightarrow \frac{1}{x+1} = 1 - \frac{20}{21} \Rightarrow x = 20$.

12. **55** Factor and solve: $x^3 + 11x^2 - 25x = 275 \Rightarrow$
$x^2(x+11) - 25(x+11) = 0 \Rightarrow (x^2 - 25)(x+11) = 0$. The three solutions are $x = 25$, -15, and -11. $(-11)(-25) = 275$.

13. **648** The figure is a square with vertices at $(\pm 18, 0)$ and $(0, \pm 18)$. A side is $18\sqrt{2}$ and the area is 648.

14. **5** The formula for the length of a diagonal of a rectangular prism is $d = \sqrt{a^2 + b^2 + c^2}$ so $c = \sqrt{d^2 - (a^2 + b^2)} = \sqrt{121 - (68 + 28)} = 5$.

Team Problem Solving - NMT 2005 **Solutions**

16. **14** Tom and Kevin's combined rate is 7.5 mph. They meet every .5 miles. The time it takes to meet is found using $t = \frac{d}{t} = \frac{.5}{7.5} = \frac{1}{15}$ hr. That means every 4 minutes they pass each other so in 1 hour they pass each other 16 times. But the initial start and the final meeting are not in the time interval.

17. **16** Multiply row one by 2 and row three by 3 and then add all three equations to get the result: $7(A + B + C + D) = 112$ so $A + B + C + D = 16$.

18. **906** Apply the formula $A = p(1+r)^t$ so $1{,}664{,}775 = p(1.05)^2 \Rightarrow$
$p = \frac{1{,}664{,}775}{1.05^2} = 1{,}510{,}000$ and $1{,}510{,}000 = 1.51 \times 10^6$. It follows that 100(1.51)(6) = 906

19. **25** The statement is true when $x^2 - 4x - 165 < 0 \Rightarrow$
$(x - 15)(x + 11) < 0 \Rightarrow -11 < x < 15$. There are 25 integers that satisfy this last inequality.

20. **32** Notice that $511 = 512 - 1 = 2^9 - 1$. Apply long division to find the remainder, which is $2^5 = 32$.

$$\begin{array}{r|l} & 2^{14} - 2^5 \\ \hline 2^9 - 1 & 2^{23} \\ & 2^{23} - 2^{14} \\ \hline & 2^{14} \\ & 2^{14} - 2^5 \\ \hline & 2^5 \end{array}$$