Grade Level 9 - NMT 2006 **Solutions**

1. $\mathbf{4n^2 + 20n}$

   $(2n+5)^2 - 25 = 4n^2 + 20n + 25 - 25$.

2. **49** $\sqrt{36 \cdot 3} + \sqrt{36 \cdot 2} = 6\left(\sqrt{3} + \sqrt{2}\right)$. The required answer is given by $6^2 + 3^2 + 2^2$.

3. **23** In a triangle, the sum of the lengths of any two sides must be greater than the length of the third side. Thus, (17 – 12) < x < (17+12).

4. $\boldsymbol{k = \frac{x}{5}}$ The measure of the supplement of the complement of an acute angle whose measure is x can be represented as 180 – (90 – x) = 90 + x. So $x = 5k$.

5. **324** $81x^4 - 16 = \left(9x^2 + 4\right)\left(9x^2 - 4\right) = \left(9x^2 + 4\right)(3x + 2)(3x - 2)$. So $a = 9$, $b = 4$, $c = 3$, and $d = 2$. The required answer is the result of evaluating the expression $(9 + 4 + 3 + 2)^2$.

6. $\mathbf{\frac{-5}{4}}$ The quadratic expression $12x^2 - 17x - 40$ factors as $(3x - 8)(4x + 5)$. Therefore the roots of the equation are $\frac{8}{3}$ and $-\frac{5}{4}$. An alternative approach would be to use either the sum or product of the roots theorems.

7. **– 17** Rearrange the terms so that conjugates are next to each other. $\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right) = -1$ and $\left(3\sqrt{2}+1\right)\left(3\sqrt{2}-1\right) = 17$.

8. $\mathbf{12\sqrt{3}}$ The area of an equilateral triangle is $A = \frac{s^2\sqrt{3}}{4}$. So $\frac{s^2\sqrt{3}}{4} = 12\sqrt{3} \Rightarrow s^2 = 48 \Rightarrow s = 4\sqrt{3}$. The perimeter is $3s = 12\sqrt{3}$.

9. **5** $\dfrac{2^{-2} + 3^{-2}}{4^{-2}} = \dfrac{\frac{1}{4} + \frac{1}{9}}{\frac{1}{16}} = \dfrac{36 + 16}{9} = 5\frac{7}{9}$. Since n is an integer, $n = 5$.

10. **50** Note that $2(2x - 7y) + 7x + 10y = 11x - 4y$ therefore $11x - 4y = 2(10) + (30) = 50$.

11. $\mathbf{\frac{5}{12}}$ The possible prime sums are 2, 3, 5, 7, and 11. The probability of rolling each of these results are: $P(2) = \frac{1}{36}$, $P(3) = \frac{2}{36}$, $P(5) = \frac{4}{36}$, $P(7) = \frac{6}{36}$, and $P(11) = \frac{2}{36}$. Therefore $P(\text{prime}) = \frac{15}{36}$.

12. **62** Either recognize the Pythagorean triple 7, 24, 25 or apply the Pythagorean Theorem.

13. **54** The lines intersect at (0,0), (9,6), and (9,–6) so the base of the triangle is 12 and its height is 9.

14. **13** Square both sides of the equation and then cube both sides.

15. **6** Let t = Curt's time and r = Curt's rate. Solve the equation $rt = 1.2r(t-1)$.

Grade Level 10 - NMT 2006 **Solutions**

1. $\mathbf{\frac{-3}{4}}$ In slope-intercept form, the equation of the second line is $y = \frac{8}{a}x + \frac{3}{a}$. When two lines are parallel, their slopes are equal. Therefore, $\frac{8}{a} = -2$ and $a = -4$. The y-intercept is $\frac{3}{a}$.

2. **18:6:25**

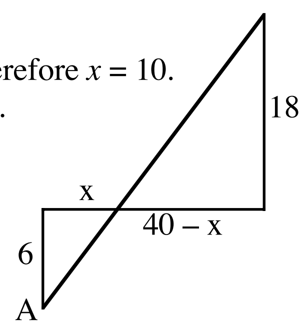
The degree-measure of an interior angle of a regular polygon of n sides is given by $\frac{180(n-2)}{n}$ and the degree-measure of an exterior angle of a regular polygon of n sides is given by $\frac{360}{n}$. An interior angle of a regular pentagon measures 108°, an exterior angle of a regular decagon is 36°, and an interior angle of a regular dodecagon measures 150°.

3. **600** ΔJKL is a 30-60-90 triangle. $\overline{KP}$ divides ΔJKL into two triangles. ΔJKP is a 30-60-90 triangle. $JK = 5\sqrt{3}$, $JL = 15$, and $KL = 10\sqrt{3}$.

4. **4** Let r and $4r$ be the two roots. Applying the sum and product rules for quadratics, $5r = \frac{-15}{k}$ and $k = \frac{-3}{r}$. $4r^2 = \frac{9}{k} = \frac{9}{-3/r} = -3r$. Solving, results in $r = 0$ or $r = \frac{-3}{4}$. Since $k = \frac{-3}{r}$, $k = 4$

5. **30** The slope of the line $3x + 5y = 30$ is $\frac{-3}{5}$, its y-intercept is 6 and its x-intercept is 10. The line $y = 6$ is horizontal and the line $x = 10$ is vertical. The region bounded by the graphs of the three lines is a right triangle whose legs measure 6 and 10.

6. **240** The two resulting triangles are similar with a ratio of similitude of 3:1. Therefore $x = 10$. The difference in their areas = 270 – 30 = 240. See the diagram at the right.

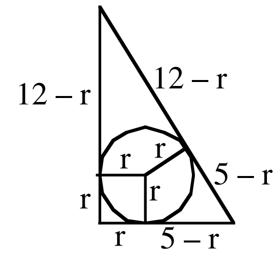


7. **11** The slope of $\overleftrightarrow{AB} = \frac{1}{2}$ and the midpoint of $\overline{AB}$ is (6, –1). Therefore the equation of the perpendicular bisector of $\overline{AB}$ is $y - (-1) = -2(x - 6)$ and the y-intercept = 11.

8. **6** This is a problem that may be done using trial and error, or by converting 70 base 10 into an equivalent number in base 3. This is done by dividing 70 by the greatest power of 3 that yields an integer quotient (3) and a remainder. Then divide by the descending powers of 3 until no remainder exists. $70_{10} = 2 \cdot 3^3 + 1 \cdot 3^2 + 2 \cdot 3 + 1$ so $A = 2$, $B = 1$, $C = 2$, and $D = 1$.

9. **17** The greatest value of $x - y$ is attained when x is a maximum and y is a minimum. This occurs when $x = 18$ and $y = 1$.

10. **2** Two tangents drawn to a circle from the same exterior point are congruent. Therefore, as can be seen in the diagram, the following equation may be set up and solved for r: $(12 - r) + (5 - r) = 13$. [Note: there is a pattern relating the value of r to the Pythagorean triples … when $r = 1$, 3-4-5; when $r = 2$, 5-12-13; etc.]



11. $\mathbf{\dfrac{20}{27}}$ This falls into the category of a Bernoulli experiment. The probability that it rains at least two out of three consecutive days is the sum of the probabilities it rains exactly two consecutive days plus the probability it rains on all three days. The formula for this is

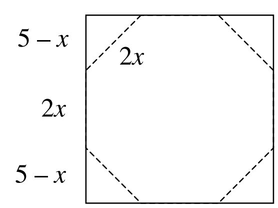
$$P(\text{rain}) = {}_3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) + {}_3C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0.$$

12. **421** The least common multiple of the six values plus 1 results in a remainder of 1 when divided by each of the integers. The LCM of 2, 3, 4, 5, 6, and 7 is 420.

13. **2** The sum of the given expressions is $2x^2 + 9x - 8$ and this equals 60. Therefore $x = 4$ which determines a mode of 4 and a median of 6. Consequently their positive difference is 2.

14. **8** Since $24 + x^2 = y^2$, $24 = y^2 - x^2 = (y - x)(y + x)$. The only possible factors of 24 that result in integer values for x and y are (12)(2), (–12)(–2), (6)(4), and (–6)(–4). Hence the only ordered pairs that satisfy the equation are $(\pm 5, \pm 7)$ and $(\pm 1, \pm 5)$.

15. **(10, 5)** Label the sides of the octagon $2x$ and the legs of the isosceles triangle are each $5 - x$. Consider one of the right isosceles triangles and solve the equation $(5 - x)^2 + (5 - x)^2 = (2x)^2$. Since $x = -5 + 5\sqrt{2}$ and $5 - x = 10 - 5\sqrt{2}$, so $a = 10$ and $b = 5$. Alternate solution: If each leg of the isosceles triangle measures x, its hypotenuse measures $x\sqrt{2}$ and the side of the regular octagon measures $10 - 2x$. Equating the two expressions and solving results in $x = 10 - 5\sqrt{2}$.



Grade Level 11 - NMT 2006 **Solutions**

1. **(2,–10)**

   Since the two linear functions are symmetric to the line $y = x$, each is the inverse of the other. Solving the equation $x = \frac{1}{2}y + 5$ for y yields $y = 2x - 10$.

2. $\mathbf{\frac{11}{6}}$ $6x^2 + 7x > 3 \Rightarrow 6x^2 + 7x - 3 > 0 \Rightarrow (3x-1)(2x+3) > 0 \Rightarrow x > \frac{1}{3}$ or $x < -\frac{3}{2}$. The length of the interval is $\left|\frac{1}{3} - \left(-\frac{3}{2}\right)\right| = \frac{11}{6}$.

3. **(836, –25, 100)**

   The amplitude, frequency, and period for $y = a\cos(bx)$ are $|a|, |b|,$ and $\frac{2\pi}{|b|}$ respectively. Thus $a^2 + b - p = \left(\frac{3}{5}\right)^2 + 8 - \frac{2\pi}{8} = \frac{836 - 25\pi}{100}$.

4. $\mathbf{18\pi - 27\sqrt{3}}$

   The region in question is called a segment of a circle. The measure of the central angle and its intercepted arc are each $\frac{1}{6}$ the measure of the circle, so the area of the sector bounded by the sides of the central angle and the arc is 18π. The area of $\Delta AOB = \frac{1}{2} \cdot 6\sqrt{3} \cdot 9 = 27\sqrt{3}$. The required area is the difference between the area of the sector and the area of the triangle. [Note: The area of $\Delta AOB = \frac{1}{2} \cdot a \cdot b \cdot \sin C = \frac{1}{2} \cdot r \cdot r \cdot \sin\frac{\pi}{3} = \frac{1}{2} \cdot r^2 \cdot \frac{\sqrt{3}}{2} = 27\sqrt{3}$.]

5. **(*z, a, x, y*)**

   $7^3 = y,\ \sqrt{x} = 3^2 = 9,\ z^3 = 8,$ and $4^a = 64$. So $y = 343,\ x = 81,\ z = 2,$ and $a = 3$.

6. **37** Solve $x^2 + (3x-1)^2 = (3x+1)^2$, so $x = 12$ and the hypotenuse is 37.

7. **15** $f(4) = 5$, so $f(16) = f(4 \cdot 4) = f(4) + f(4) = 5 + 5 = 10$, and $f(64) = f(4 \cdot 16) = f(4) + f(16) = 5 + 10 = 15$.

8. **1** For a non-zero sequence to be both arithmetic and geometric all the terms must be the same. Therefore the ratio of consecutive terms is **1**.

9. **0** In the given equation, $5^{2x+1} + 5 = 26 \cdot 5^x$, let $u = 5^x$. Therefore $5^{2x+1} = 5 \cdot (5^x)^2 = 5u^2$. The new equation is $5u^2 + 5 = 26u$. Factor and solve to get $u = 5$ and $u = \frac{1}{5}$. Thus $x = \pm 1$.

10. **49** Factor $a^2b + ab^2 + a + b = 77$, so that $(ab+1)(a+b) = 77$. Substitute $ab = 10$, so that $a + b = 7$. Therefore $(a+b)^2 = \mathbf{49}$.

11. **60** Substitute each of the following identities: $\sin(157°)=\sin(23°)$, $\sin(53°)=\cos(37°)$ and $\sin(102°)=\sin(78°)=\cos(12°)$, $\cos(138°)=-\cos(42°)=-\sin(48°)$ into the original expression

$$\begin{aligned}\frac{\sin(157°)\sin(53°)+\cos(23°)\sin(37°)}{\sin(102°)\cos(48°)+\sin(12°)\cos(138°)} &= \frac{\sin(23°)\cos(37°)+\cos(23°)\sin(37°)}{\cos(12°)\cos(48°)-\sin(12°)\sin(48°)} \\ &= \frac{\sin(23°+37°)}{\cos(12°+48°)} \\ &= \frac{\sin(60°)}{\cos(60°)} \\ &= \mathbf{\tan(60°)}\end{aligned}$$

12. **4** Square the expression $\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}}$. This results in the number 16, and then square root the result to get 4.

13. **2** $g(x)=f(x-2)+3$, so $g(5)=f(3)+3=2^3-3^2+3=2$.

14. **2 – 2*c*** $\log_{10}25=\log_{10}\frac{100}{4}=\log_{10}100-\log_{10}4=\log_{10}100-2\log_{10}2=2-2c$.

15. **– *i*** Evaluate both numerator and denominator by recognizing that $i+i^2+i^3+i^4=i-1-i+1=0$, so the numerator results in $i-1$ and the denominator is $-i-1$. Multiplying the numerator and denominator by $-i+1$ results in $\boldsymbol{-i}$.

Grade Level 12 - NMT 2006 **Solutions**

1. $\mathbf{2\sqrt{2}}$

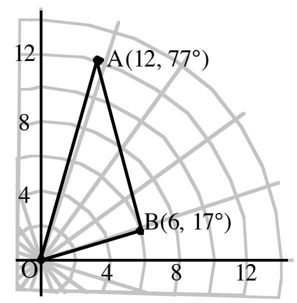
$$\lim_{x\to\sqrt{2}}\frac{x^2-2}{x-\sqrt{2}}=\lim_{x\to\sqrt{2}}\frac{x^2-2}{x-\sqrt{2}}\cdot\frac{x+\sqrt{2}}{x+\sqrt{2}}=\lim_{x\to\sqrt{2}}\frac{\left(x^2-2\right)\left(x+\sqrt{2}\right)}{x^2-2}=\lim_{x\to\sqrt{2}}\left(x+\sqrt{2}\right)=2\sqrt{2}.$$

2. **(–1, 17, 2)**

$a=\frac{1}{2}\cdot\frac{ar+ar^2}{2}$. Multiplying both sides by 4 and arranging the terms in order we get $r^2+r-4=0$. Applying the quadratic formula yields $r=\frac{-1\pm\sqrt{17}}{2}$.

3. **0** $f'(x)+g'(x)=\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}\left[\sin^2 x+\cos^2 x\right]=\frac{d}{dx}[1]=0$. Alternate solution: $f'(x)=2\sin x\cdot\cos x$ and $g'(x)=-2\sin x\cdot\cos x$, so their sum is 0.

4. $\mathbf{x=\frac{3}{2}}$ Subtract the two equations. The result is $-6x=-9\Rightarrow x=\frac{3}{2}$.



5. $\mathbf{6\sqrt{3}}$ Refer to the diagram. Since $OA{:}OB = 2{:}1$ and $m\measuredangle AOB=60°$, ΔAOB is a 30°-60°-90° triangle. Thus $AB=6\sqrt{3}$. An alternate approach is to apply the law of cosines so that $AB^2=6^2+12^2-2(6)(12)\cos(77°-17°)$.

6. $(\mathbf{2,1})$ Since $f(x)=\frac{2}{1-x}$, $f(g(x))=\frac{2}{1-g(x)}=\frac{2x+2}{x-1}$. Solving for *g(x)*, we get $g(x)=\frac{2}{1+x}$.

7. **{2, 3, 4}**

The domain of $r(n)=\frac{n^2-25}{n^3-n}$ is the set {n: n ≠ –1, 0, 1} and the domain of $s(n)=\frac{\sqrt{4-n}}{\sqrt{n+1}}$ is the set {n: n = 0, 1, 2, 3, 4} therefore the intersection of the domains is {2, 3, 4}

8. **–sin *x*** The derivatives of the sine function form a pattern of period four. Since $2006\equiv 2\pmod 4$, $\frac{d^{2006}y}{dx^{2006}}=\frac{d^2y}{dx^2}=-\sin x$.

9. **–20** Substituting $y = -5$ into the given equation and solving for x yields the value $x = 2$. Thus, the line is tangent to the curve at the point (2, –5). Differentiating implicitly, $x^3\frac{dy}{dx}+y\left(3x^2\right)+2y\frac{dy}{dx}=\frac{dy}{dx}$, and plugging in (2, –5) gives: $8\frac{dy}{dx}-60-10\frac{dy}{dx}=\frac{dy}{dx}$, so $3\frac{dy}{dx}=-60$ and $\frac{dy}{dx}=-20$.

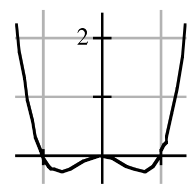
10. **37** Since the graph is tangent to the x-axis, the value $x = 2$ must be a double root of $x^3 + ax^2 + bx - 148 = 0$. The product of the roots is 148, therefore the third root is $\frac{148}{2 \cdot 2} = 37$.

11. **3** Let $P = xy$. We know that $x = 12 - ky$, so $P = (12 - ky)y = 12y - ky^2$. Since P has a maximum at $y = 2$, $\frac{dP}{dy} = 0$ when $y = 2$. Thus, $\frac{dP}{dy} = 12 - 12ky = 0$ at $y = 2$. Solve for k to find $k = 3$.

12. **0** Consider the series as the difference of two geometric series. The first has $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and the second has $a = \frac{2}{3}$ and $r = \frac{1}{3}$. Therefore the sum of the series is $\frac{\frac{1}{2}}{1-\frac{1}{2}} - \frac{\frac{2}{3}}{1-\frac{2}{3}} = 1 - 1 = 0$.

13. $\mathbf{y = -\frac{1}{4}}$

$y = x^4 - x^2 = x^2(x+1)(x-1)$, so the curve has three x-intercepts, at $x = 0$, $x = -1$, and $x = 1$. Since as x goes to either $\pm\infty$, y goes to $+\infty$ and since the function is even, the graph of the function must look like the picture at the right. The common tangent to the curve must be at the relative minima of the function. Since $\frac{dy}{dx} = 4x^3 - 2x = 0$, $x = 0$ or $x = \pm\sqrt{\frac{1}{2}}$. Using the latter 2 values in the original function yields $y = -\frac{1}{4}$.



14. $\mathbf{\frac{1}{x}}$

$$h(x) = \ln|\sin x| + \ln|x \cdot \cot x| + \ln|\sec x|$$

$$h'(x) = \frac{1}{\sin x}(\cos x) + \frac{1}{x \cdot \cot x}\left(-x \cdot \csc^2 x + \cot x\right) + \frac{1}{\sec x}\sec x \cdot \tan x$$

$$= \cot x - \frac{\sin x}{\cos x} \cdot \frac{1}{\sin^2 x} + \frac{1}{x} + \tan x = \frac{1}{x} + \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} - \frac{1}{\sin x \cdot \cos x}$$

$$= \frac{1}{x} + \frac{\sin^2 x + \cos^2 x - 1}{\sin x \cdot \cos x} = \frac{1}{x}$$

15. **11, 20, 27, 32, 35**

Since $\frac{d}{dx}\left(k - x^2\right) = -2x$, the tangent to the curve at $\left(a, k - a^2\right)$ has a slope of $-2a$. Furthermore $-2a = \frac{\Delta y}{\Delta x} = \frac{k - a^2}{a - 6}$ we find $a^2 - 12a + k = 0$. $\left(a, k - a^2\right)$ is a lattice point so a is an integer and $a^2 - 12a + k = (a - p)(a - q)$ where p and q are integer solutions of the equation. Since the point of tangency is in the first quadrant, and the line of tangency contains (6, 0), $0 < a < 6$. Without loss of generality, $0 < p < 6$. Because the sum of the roots is 12 and since k is the product of the roots, the only values for p and q are (p, q): ((1,11), (2,10), (3,9), (4, 8), and (5, 7).

1. **C** Let a, b, and c be the three integers. Then $a + b = 31$, $b + c = 90$, and $a + c = 137$. Sum all three equations together to find $a + b + c = 129$. By subtraction, all three values can be found to be 98, 39, and –8.

2. **C** The Pythagorean theorem may be used to calculate the hypotenuse of the given triangle, which is $\sqrt{40}$. Similar triangles result in the proportion, $\frac{\text{altitude}}{\sqrt{15}} = \frac{5}{\sqrt{40}}$. Solve to find the length of the altitude to be $\frac{5\sqrt{6}}{4}$. Sum the integers 5 + 6 + 4 to get 15.

3. **D** Let the rates of each machine be $\frac{1}{A}, \frac{1}{B},$ and $\frac{1}{C}$, respectively. Then solve the system of equations $\frac{2}{A}+\frac{2}{B}+\frac{2}{C}=1$, $\frac{3}{A}+\frac{3}{C}=1$, $\frac{4}{B}+\frac{4}{C}=1$ and $\frac{t}{A}+\frac{t}{B}=1$ for t.

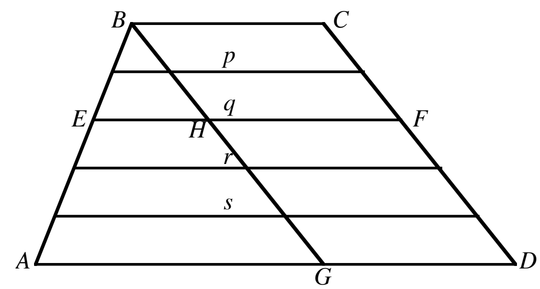
4. **C** The graphs of the equations are circles and the minimum distance between them is the distance between their centers minus the sum of their radii. The centers of the circles are (0, 0) and (–15, 8). The distance between the centers is 17 and the minimum distance between the circles is 17– (2+3).

5. **D** Since $\frac{a}{b}<1$, $\frac{b}{a}>1$. Similarly $\frac{d}{c}>1$. In addition, since $\frac{a}{b}<\frac{c}{d}$, $\frac{b}{a}>\frac{d}{c}$. Furthermore, since $\frac{d}{c}>1$, $\frac{bd}{ac}>\frac{b}{a}$. Finally $\frac{b}{a}>\frac{b+d}{a+c}>\frac{d}{c}$. Therefore, $\frac{bd}{ac}>\frac{b}{a}>\frac{b+d}{a+c}>\frac{d}{c}>1$. Alternate Solution: Replace the letters a, b, c, and d with 1, 2, 3, and 4, respectively, and test the values of the choices given.

6. **A** This is a Bernoulli experiment, so

$$\begin{aligned}
P(F) &= 1 - P(S) \\
&= 1 - \left({}_8C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^8 + {}_8C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^7 + {}_8C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^6 \right) \\
&= 1 - \left(\frac{1}{2}\right)^8 (1+8+28) \\
&= \frac{219}{256} \\
&= 0.855
\end{aligned}$$

7. **D** Draw a line parallel to $\overline{CD}$ through point B, intersecting $\overline{AD}$ at G and $\overline{EF}$ at H. Since $\Delta BEH \sim \Delta BAG$, the ratio of their altitudes equals the ratio of their bases, so $\frac{2}{5} = \frac{EH}{AG}$. Since AG = 20, EH = 8 and EF = 14+8 = 22.



Alternate Solution #1: Since the segments are equally spaced, their lengths form an arithmetic sequence, starting with 14 and ending with 34. Therefore the lengths of the segments which are parallel to the bases of the trapezoid must be 18, 22, 26, and 30.

Alternate Solution #2: Draw $\overline{BD}$ and sum the values $\frac{2}{5}(34)+\frac{3}{5}(14)$ to get 22.

8. **E** Since $G = 1+\cfrac{1}{1+\cfrac{1}{1+\ldots}} = 1+\frac{1}{G}$ and $R=\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}=\sqrt{1+R}$, each equation reduces to basically the same equation $G^2 = G+1$ and $R^2 = R+1$. The solution to each equation is the golden ratio $G = R = \frac{1+\sqrt{5}}{2}$. Thus, the product, $G \cdot R = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$.

9. **B** Examine the prime factors of the products of two integers and N divided by the third integer. So $\frac{30 \cdot 36 \cdot N}{50} = \frac{2^3 \cdot 3^3 \cdot 5 \cdot N}{2 \cdot 5^2} \Rightarrow 5$ is a factor of N. Similarly, $\frac{30 \cdot 50 \cdot N}{36} = \frac{2^2 \cdot 3 \cdot 5^3 \cdot N}{2^2 \cdot 3^2} \Rightarrow 3$ is a factor of N. $\frac{36 \cdot 50 \cdot N}{30} = \frac{2^3 \cdot 3^2 \cdot 5^2 \cdot N}{2 \cdot 3 \cdot 5}$ so no additional factors are required to satisfy the conditions. The minimal value for N is 15.

10. **B** Since $\frac{1}{a+c} = \frac{1}{a}+\frac{1}{c}$, $a^2+2ac+c^2 = ac$ and $a^2+ac+c^2=0$. Use the quadratic formula to solve for a in terms of c. This results in $a = \frac{-c \pm c\sqrt{-3}}{2}$, so $\frac{a}{c} = \frac{-1 \pm \sqrt{-3}}{2}$. Cube each side so $\left(\frac{a}{c}\right)^3 = 1$.

Alternate Solution: Since $a^2+ac+c^2=0$, multiply each side by $a-c$ to get $a^3-c^3=0$, which means that $\frac{a^3}{c^3} = \left(\frac{a}{c}\right)^3 = 1$.

Team Problem Solving - NMT 2006 **Solutions**

1. **252** $7^{3x} = \left(7^x\right)^3 = 6^3$. Thus, $7^x = 6$ and $7^{2x+1} = \left(7^x\right)^2(7) = (36)(7) = 252$.

2. **{-1, -2}** Let $a = x^2 + 3x + 7$. Then, $a^2 - 15 = 2a$, and a = 5 or –3. If $5 = x^2 + 3x + 7$, then $x = -1$ or –2. If $-3 = x^2 + 3x + 7$, then x is not real.

3. **(7, 31)** If *v, w, x, y,* and *z* are the five integers, then $4(v + w + x + y + z) = 60 + 68 + 72 + 80 + 84 = 364$. Therefore, the sum of the five integers is 91. The least integer has the value 91 – 84 and the greatest integer has the value 91 – 60.

4. **16** By the Pythagorean Theorem, the top of the ladder is 96 inches up the wall. When the ladder is [28+4(8)] or 60 inches from the base of the wall, the top of the ladder is 80 inches up the wall.

5. **$1,749,600**
   The company's sales in **2003** were $\$17{,}006{,}112(1.08)^{-2}$ or exactly $14,580,000. Its profit for that year is given by $14,580,000(0.12).

6. **–33** The quantity, $n^2 - 16n + 105 = n^2 - 16n + 64 + 41 = (n-8)^2 + 41$, which is divisible by 41 when $n - 8$ is divisible by 41.

7. **9** The units digit in the expansion of 2007^n, where n is a nonnegative integer is cyclic of order 4. For n = 0, 1, 2, and 3, the units digit in each of the expansions is 1, 7, 9, and 3 respectively. 2006/4 = 501 with remainder 2.

8. **(90,54,36)**
   A third equation that must be satisfied is $x + y + z = 180$. By solving the system of three equations, $(x,y,z) = (90,54,36)$.

9. **34** The graphs of $20x + 6y = 2006$ and $10x + 3y = 1003$ are coincident. Solving for y in the second equation yields $y = \dfrac{1003 - 10x}{3}$, which is a positive integer when x = 100, 97, 94, … , 1.

10. **280** When Alan crosses the finish line, Bob is at the 1430-yard line and Charles is at the 1210-yard line. Let x be the yard line number where Charles is when Bob finishes the race. Since they run at uniform rates, for any given time the ratio of their distances is constant so, $\dfrac{1430}{1210} = \dfrac{1820}{x}$ and x = 1540. 1820 – 1540 = 280.

11. **5** $A = B - 1$ and $C = B + 1$. squaring both sides of the equation, $(B-1)^2 + B^2 + (B+1)^2 = (3B)^2$. This yields the three possible values of B as 0, 1, or 4. The values 0 and 1 are ruled out by the conditions of the problem. Since B = 4, C = 5.

12. **21** The formula for finding the length of a diagonal, d, of a rectangular prism, $d = \sqrt{l^2 + w^2 + h^2}$, is easily derived using the Pythagorean Theorem. Substituting in the formula reveals the third dimension to be 21.

13. $\mathbf{\frac{3}{50} = 0.06}$

Let the two numbers be m and n. Then, $\frac{1}{m^2}+\frac{1}{n^2}=\frac{m^2+n^2}{(mn)^2}$. Now, $m^2+n^2=(m+n)^2-2mn$, which is 64 – 40 or 24 and $(mn)^2=400$. Thus, the sum of the reciprocals of the squares is $\frac{24}{400}$.

14. **{–5, –1, 0, 1, 3}**

By observation, the equation is satisfied when $x = -1$, 0, or 1. When $x \neq 0$, $(x+1)^2=16$; $x = 3$ or –5.

15. $\mathbf{\frac{1}{4}}$ In order of size, the elements of S are $\{1, x, y, x+y\}$. The mean of S is $\frac{2x+2y+1}{4}=\frac{x}{2}+\frac{y}{2}+\frac{1}{4}$. The median of S is $\frac{x+y}{2}=\frac{x}{2}+\frac{y}{2}$.

16. **200** When A finishes the race, the ratio of the distances apart of B to C is $\frac{d-40}{d-56}$. When B finishes the race, the ratio of the distances apart of B to C is $\frac{d}{d-20}$. Since these two ratios are equal, $d^2-56d=d^2-60d+800$. So $d = 200$.

17. **9** Letting the merchant's cost be c, it follows that $36(1.5c) = (36 + x)(1.2c)$.

18. $\left\{\left(\frac{1}{\sqrt[3]{2}},-3\right),\left(\sqrt{2},2\right),\left(-\sqrt{2},2\right)\right\}$

$(2x)^{y^2}=\left(2^{y^2}\right)\left(x^{y^2}\right)=64$; $\left(2^{y^2}\right)\left(x^{y}\right)^{y}=64$; $\left(2^{y^2}\right)\left(2^{y}\right)=2^6$; $y^2+y=6$, so $y = -3$ or 2.

19. **19** Let Sheila's age = s, then Art's age = s+17. Let 100s+(s+17) = y^2 and 100(s+13)+(s+30) = x^2. Therefore $x^2-y^2=1313$. Since x and y are 2-digit integers satisfying $x^2-y^2=1313$, it follows that $(x+y)(x-y)=(101)(13)$. Solving the pair of equations, $x + y = 101$ and $x - y = 13$ yields $y = 44$ so $y^2=1936$. Therefore, Sheila's age is the first two digits of this square, 19.

20. **200.6** Let S = the sum. Then $9S=2006-\frac{2000}{9}+\frac{2006}{9^2}-\frac{2006}{9^3}+\ldots$. Adding, $10S = 2006$ so $S=200.6$.