Nassau County Interscholastic Mathematics League

Team Contest   Answers must be integers from 0 to 999, inclusive.   2023 – 2024

Calculators are allowed.

Time: 40 minutes

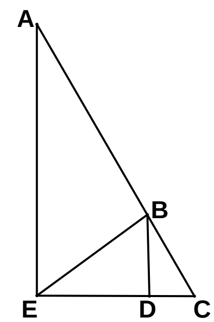
31. The vertex of the parabola whose equation is $y = x^2 - 20x + c$ is on the x-axis. Compute c.

32. Compute the sum of the two positive integer values of x that satisfy $xy + 4x - 13 = x^2$.

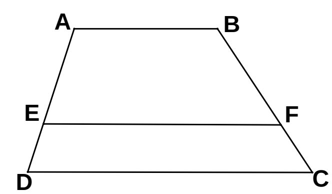
33. The x-coordinate of the point P on the line whose equation is $4x + 3y = 12$ that is closest to the origin may be written in simplest form as $\frac{p}{q}$. Compute $p + q$.

34. Six hundred apples were harvested and lined up in a single row. When inspected in the row order, it was found that every third apple was too small, every fourth apple was too green, and every tenth apple was bruised. The rest of the apples were perfect. Compute the number of harvested apples that were perfect.

35. In the accompanying diagram, in right triangle AEC, $\measuredangle E$ is a right angle, $\overline{EB} \perp \overline{ABC}$ at point B. and $\overline{BD} \perp \overline{EDC}$ at point D.
If $AB = 48$, $BC = 16$, and $BD = p\sqrt{q}$ in simplest form, compute $p + q$.



36. In trapezoid $ABCD$, $\overline{AB} \parallel \overline{CD}$. Points E and F are on $\overline{AD}$ and $\overline{BC}$ respectively, so that $\frac{AE}{ED} = \frac{BF}{FC} = 2$. If $AB = 7$ and $DC = 22$, compute EF.



37. In triangle ABC, $AB = 30$, $BC = 36$, and $AC = 60$. Three circles have their centers at points A, B, and C. These three circles are externally tangent in pairs. Compute the radius of the largest of the three circles.

38. If $x + y + z = 12$, $xy + yz + xz = 29$, and $xyz = 18$, compute the maximum possible value of x.

39. If a, b, and c are greater than 1 and $\log_a b = 4$ and $\log_b c = \frac{1}{8}$, compute $\log_c \sqrt{ab}$.

40. Let $x \geq 2$, $f_0(x) = \frac{1}{1-x}$, and for $n = 1, 2, 3, \ldots$, let $f_n(x) = f_0\big(f_{n-1}(x)\big)$.
Compute $f_{602}(158)$.

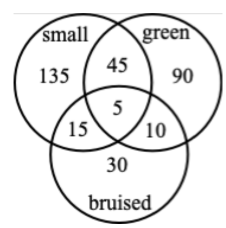
Solutions for Team Contest

31. Since the coordinates of the vertex of the parabola are $\left(\frac{-b}{2a}, 0\right) = (10, 0)$, an equation of the parabola is $y = (x - 10)^2 = x^2 - 20x + 100$. Thus $c = \mathbf{100}$. Alternatively, from the given, $x^2 - 20x + c = 0$ has equal roots. Thus, $b^2 - 4ac = 0 \to 400 - 4c = 0 \to c = 100$.

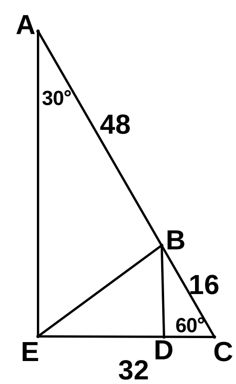
32. From the given equation, $xy + 4x - x^2 = 13 \to x(y + 4 - x) = 13$. Since 13 is prime and x is a positive integer, $x = 1$ or $x = 13$. In either case, $y = 10$. The required sum is **14.**

33. The distance from a point to a line is the length of the perpendicular segment from the point to the line. The slope of the given line is $-\frac{4}{3}$. The equation of the perpendicular line segment through the origin is $y = \frac{3}{4}x \to \frac{3}{4}x = -\frac{4}{3}x + 4 \to 9x = -16x + 48 \to 25x = 48 \to$ $x = \frac{48}{25}$ and the required sum is **73.**

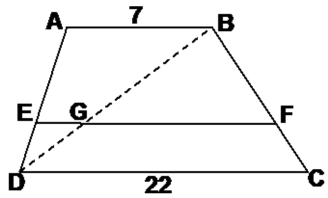
34. One third of the apples were too small. One fourth of the apples were too green. One twelfth of the apples were too small and too green. One tenth of the apples were bruised. One thirtieth of the apples were bruised and too small. One fortieth of the apples were bruised and too green. One one-hundred twentieth of the apples were bruised and too small and too green. The fraction of the harvested apples that were not perfect is $\frac{1}{3} + \left(\frac{1}{4} - \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{30} - \frac{1}{40} + \frac{1}{120}\right) = \frac{1}{3} + \frac{1}{6} + \frac{1}{20} = \frac{33}{60} = \frac{11}{20}$.
Therefore, $\frac{9}{20} \cdot 600 = \mathbf{270}$ were perfect. Alternatively, using the numbers of apples, $200 + (150 - 50) + (60 - 20 - 15 + 5) = 330$, and $600 - 330 = 270$.
The Venn diagram shows all of the apples that were imperfect.



35. First, $\Delta AEC \sim \Delta EBC \sim \Delta BDC \to \frac{AC}{EC} = \frac{EC}{BC} \to \frac{64}{EC} = \frac{EC}{16} \to EC = 32$. Since $AC = 64$ and $EC = 32$ in right ΔAEC, then $m < A = 30°$ and $m < C = 60°$. Then, by similar triangles, ΔEBC and ΔBDC are also 30-60-90 triangles. So, if $BC = 16$, then $BD = 8\sqrt{3}$ and the required sum is **11.**



36. Draw $\overline{DB}$ to intersect $\overline{EF}$ at G. So, $\Delta BGF \sim \Delta BDC \to \frac{GF}{DC} = \frac{BF}{BC} \to \frac{GF}{22} = \frac{2}{3} \to$ $GF = \frac{44}{3}$. Similarly, $\Delta DEG \sim \Delta DAB \to \frac{EG}{AB} = \frac{DE}{DA} \to \frac{EG}{7} = \frac{1}{3} \to EG = \frac{7}{3}$. Thus, $EF = \frac{44}{3} + \frac{7}{3} = \mathbf{17}$.



37. The segment joining the centers of two tangent circles contains the point of tangency. Call the points of tangency D, E, and F, with $\overline{ADB}$, $\overline{BEC}$, and $\overline{CFA}$. Then, let $AD = x = AF, BD = y = BE$, and $CE = z = CF$. Then, $x + y = 30$, $y + z = 36$, and $x + z = 60 \to 2x + 2y + 2z = 126 \to x + y + z = 63 \to z = \mathbf{33}$.

38. Since the sum of the roots is 12, the product of the roots is 18, and the sum of the product of the roots taken two at a time is 29, then x, y, and z are roots of $a^3 - 12a^2 + 29a - 18 = 0 \to (a-1)(a-2)(a-9) = 0$. So, $\{x, y, z\} = \{1,2,9\}$. The solutions of the given system are permutations of $(1,2,9)$. The maximum value of x is **9**. Alternatively, $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + xz) = 144 \to$ $x^2 + y^2 + z^2 + 58 = 144 \to x^2 + y^2 + z^2 = 86$. Therefore, $y = 1, z = 2$, and $x = 9$.

39. From the given equations, we deduce that $\frac{\log b}{\log a} = 4$ and $\frac{\log c}{\log b} = \frac{1}{8} \to \frac{\log c}{\log a} = \frac{1}{2} \to$ $\frac{\log a}{\log c} = 2 \to \log_c a = 2$. Also, $\frac{\log b}{\log c} = \log_c b = 8$. Then, $\log_c \sqrt{ab} = \frac{1}{2}(\log_c a + \log_c b) =$ $\frac{1}{2}(2 + 8) = \mathbf{5}$.

40. From the given information, $f_1(x) = f_0(f_0(x)) = \frac{1}{1 - \frac{1}{1-x}} = \frac{1-x}{-x} = \frac{x-1}{x}$ and $f_2(x) =$ $f_0(f_1(x)) = \frac{1}{1 - \frac{x-1}{x}} = \frac{x}{x-x+1} = x$ and $f_3(x) = f_0(f_2(x)) = \frac{1}{1-x} = f_0(x)$. Then, the pattern continues in cycles of 3. To determine $f_{602}(x)$, divide 602 by 3 and since the remainder is 2, $f_{602}(x) = f_2(x) = x$. Therefore, $f_{602}(158) = \mathbf{158}$.