Nassau County Interscholastic Mathematics League

Contest #2 Answers must be integers from 0 to 999, inclusive. 2016 – 2017

Calculators are allowed.

Time: 10 minutes

7) One evening, Amanda, Melanie, and Sophia gambled among themselves. At the start of the evening, they each had an identical amount of money. At the end of the evening, the amount of money they had was in the ratio of 3:4:5 in the given order. If Sophia increased her original amount of money by $x\%$, compute x.

8) Isaac chose two numbers. The positive difference of his two numbers equals the product of the two numbers. Jake chose two numbers which are the reciprocals of each of Isaac's numbers. Compute the positive value of the difference of Jake's numbers.

Time: 10 minutes

9) Compute the remainder when the smallest perfect cube that is a multiple of 175 is divided by 1000.

10) Three edges of a cube are $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$. Point E is on $\overline{AC}$ and $AE = 3EC$. If $AB = 4$ and the area of ΔDEB in simplest form is $a\sqrt{c}$, compute $a + c$.

Time: 10 minutes

11) Compute the number of <u>even</u> three-digit base 9 numbers that may be formed using each of the digits 3, 4, and 5 exactly once.

12) Ten people are playing cards with a standard deck of 52 cards. Each player receives two cards. When the probability that exactly two players each receive two aces is multiplied by 1,000,000 and rounded to the nearest integer, compute the result.

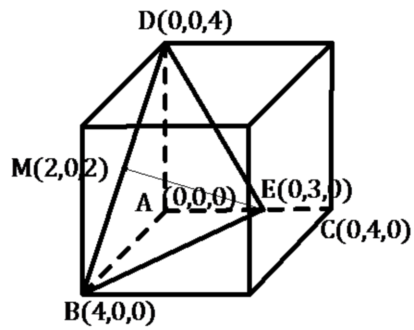
Solutions for Contest #2

7) Without loss of generality, suppose they each started with \$4. Since the amount of money in the room was \$12, at the end of the evening, Amanda had \$3, Melanie had \$4, and Sophia had \$5. Sophia's sum increased by \$1. Her sum increased by 25%, so $x = \mathbf{25}$.

8) Isaac selects numbers x and y such that $x - y = xy$, with $x > y > 0$. So, $\frac{1}{y} > \frac{1}{x}$ and Jake calculates $\frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy} = \frac{xy}{xy} = \mathbf{1}$. Alternatively, one can guess that $x = \frac{1}{2}$ and $y = \frac{1}{3}$ satisfy the conditions so that $3 - 2 = \frac{1}{6}/\frac{1}{6} = \mathbf{1}$.

9) Since $175 = 5^2 \cdot 7$, the smallest perfect cube that is a multiple of 175 is $5^3 \cdot 7^3 = 35^3 = 42875$. When 42875 is divided by 1000, the remainder is **875**.

10) Place the cube in a 3-space coordinate system with $A(0,0,0)$, $B(4,0,0)$, $C(0,4,0)$, $D(0,0,4)$, and $E(0,3,0)$. Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$, $BD = 4\sqrt{2}$, $DE = 5$, and $BE = 5$. If $M(2,0,2)$ is the midpoint of $\overline{BD}$, the base of isosceles ΔBED, then $EM = \sqrt{17}$ is the height. So, using $A = \frac{1}{2}bh$, the area of $\Delta BED = \frac{1}{2}(4\sqrt{2})(\sqrt{17}) = 2\sqrt{34}$. Thus, the required sum is **36.**



Alternatively, we can use Heron's Formula:
$A = \sqrt{s(s-a)(s-b)(s-c)}$, where the semiperimeter, $s = \frac{10+4\sqrt{2}}{2} = 5 + 2\sqrt{2}$.
So, $A = \sqrt{(5+2\sqrt{2})(5-2\sqrt{2})(2\sqrt{2})(2\sqrt{2})} = \sqrt{(25-8)(8)} = \sqrt{17 \cdot 8} = 2\sqrt{34}$, and the required sum is **36.**

11) Since 9 is odd, each of its integer powers is odd. Since, exactly one of $\{3,4,5\}$ is even, any of the six possible base 9 numbers: 345_9, 354_9, 435_9, 453_9, 534_9, 543_9, is also even. To illustrate, $345_9 = 3(9^2) + 4(9^1) + 5(9^0) = 243 + 36 + 5 = 284$, which is even. So the requirements of the problem are met by all **6** of these base 9 numbers.

12) The probability that Player #1 gets two aces is $\frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652}$. The probability that Player #2 gets two aces is $\frac{2}{50} \cdot \frac{1}{49} = \frac{2}{2450}$. The probability that any two of the ten players each get two aces is ${}_{10}C_2 \left(\frac{12}{2652}\right)\left(\frac{2}{2450}\right)$. Use a calculator to multiply this product by 1,000,000 and round to get **166.**