Practice #10 – Sequences and Series

Definitions

A **sequence** is simply an ordered list of values, such as 1, 3, 5, 7, 9, 11, …. Such a sequence can be either finite or infinite (though most are infinite) and there may or may not be a rule defining such a sequence (though it is really hard to ask questions about sequences without rules.) Such a rule can be defined two different but equivalent ways, explicitly or recursively.

An **explicit** definition is essentially a function of n, where n must be a natural number representing the position in the list. For the example above, the formula

$$a_n = 2n - 1 \qquad \text{or} \qquad \{2n-1\}_{n=1}^{\infty}$$

serves as an explicit definition for the sequence. The major advantage of an explicit definition is that it is possible to determine any single element of the sequence by simply knowing its position in the list; not so for a recursive definition.

A **recursive** definition is a way of defining a given term of a sequence in terms of the previous term or terms. For the example above, the formula

$$a_{n+1} = a_n + 2$$

serves as (part of) a recursive definition for the sequence. However, it is not sufficient. The sequence 4, 6, 8, 10, 12, 14, … is also served by this recursive definition. Since the recursive rule is based on the previous term, changing the first term of the sequence can change the whole sequence. Therefore, a proper recursive definition of the sequence above would read:

$$a_{n+1} = a_n + 2,\ a_1 = 1$$

Recursive definitions for sequences are usually easier to determine, but have the disadvantage that if one wishes to find, say, the 2009th term of the sequence, one needs to find all of the intervening terms.

A **series** is simply the sum of a series, which again may be finite or infinite. When written in explicit form, a series typically uses **sigma notation**, such as:

$$\sum_{n=1}^{7}(2n-1) = 1+3+5+7+9+11+13$$

An infinite series can also be constructed, but does not always add to a finite value. A series such as $\sum_{n=1}^{\infty}(2n-1) = 1+3+5+7+9+\ldots$ which does not add to a finite value, is said to **diverge**.

On the other hand, a series such as

$$\sum_{n=1}^{\infty} 3\left(\frac{1}{10}\right)^n = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots = 0.3 + 0.03 + 0.003 + \ldots = 0.333\ldots = \frac{1}{3}$$

which *does* add up to a finite value, is said to **converge**. It is a necessary (but not sufficient—see the section on the harmonic series) fact that in order for a series to converge, the individual *terms* of the series must be getting closer and closer to zero.

The **nth partial sum** (S_n) of a series is defined to be the sum of the first n terms of the series. It is taken as a definition that if an infinite series converges, then the **sequence of partial sums**

$$S_1, S_2, S_3, \ldots$$
$$= a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots$$

must get closer and closer to a specific, finite value, called the **limit** of the sequence.

<u>The Arithmetic Sequence and Series</u>

An **arithmetic sequence** (pronounced arithMETic, not aRITHmetic) is defined recursively as

$$\boxed{a_{n+1} = a_n + d,\ a_1 = a}$$

and therefore takes the form

$$a, a + d, a + 2d, a + 3d, \ldots$$

This pattern leads us quickly to an explicit form

$$\boxed{a_n = a + (n-1)d},$$

a formula which can easily be rederived as needed if forgotten.

An infinite **arithmetic series** will always diverge unless a = d = 0 (which is boring), so we can consider only the sum of a *finite* arithmetic series.

$$S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n-1)d)$$

Such a series can be written in reverse order without changing its value, of course:

$$S_n = (a + (n-1)d) + (a + (n-2)d) + (a + (n-3)d) + \ldots + a$$

Adding the two series term by term yields:

$$2S_n = n\left[2a + (n-1)d\right]$$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

This formula can be somewhat awkward, but if we note that $2a+(n-1)d=a_1+a_n$, the formula can be simplified to

$$\boxed{S_n=\frac{n}{2}(a_1+a_n)}$$

An even simpler formula can be used if the sequence has an odd number of terms. Note that

$$S_n=\frac{n}{2}\left[2a+(n-1)d\right]=n\left[a+\frac{n-1}{2}d\right]$$

$$\boxed{S_n=na_{\frac{n-1}{2}}}$$

proving that the sum of an arithmetic series is equal to the product of the number of terms and the middle term of the series. As a corollary, this shows that the mean and the median of an arithmetic series are the same. This can also be shown to be true if the number of terms in the series is even (prove it!)

One last tactical note: in math team problems involving arithmetic series, it is often easier to refer to the middle term as a instead of the first term, especially if the terms need to be multiplied. For example, if a question involves the product of 5 consecutive even integers (an arithmetic series), if we start from the first term, the product becomes

$$a(a+2)(a+4)(a+6)(a+8)=a^5+20a^4+140a^3+400a^2+384a$$

however, starting from the middle term, we get instead

$$(a-4)(a-2)a(a+2)(a+4)=a(a^2-4)(a^2-16)=a^5-20a^3+64a$$

a shorter and easier-to-find result.

<u>The Geometric Sequence and Series</u>

A geometric sequence is defined recursively by

$$\boxed{a_{n+1}=a_n r,\ a_1=a}$$

and therefore takes the form

$$a, ar, ar^2, ar^3, \ldots$$

This pattern leads us to the explicit form

$$\boxed{a_n=ar^{n-1}}.$$

Unlike arithmetic series, an infinite **geometric series** can either converge or diverge, depending on the value of r. To determine the interval of convergence, we'll need to look at finite geometric series first.

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

We start by multiplying both sides by r:

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n$$

and then subtracting the bottom equation from the top. This cancels out *most* of the terms:

$$S_n - rS_n = a - ar^n$$

From this we get the formula

$$\boxed{S_n = a\frac{1-r^n}{1-r}}.$$

There are no significantly simpler forms of this formula.

In order to determine whether or not an infinite geometric series converges, we need to look at what happens to the value of S_n as the value of n gets larger and larger. Notice that the only part of the formula for the nth partial sum of the series that contains an n is the exponential term r^n. If the value of r is greater than one, the term will grow without bound as n increases. Similarly, if the value is less than negative one, it will grow without bound as well, but will also alternate signs as it does. In neither case does the value of S_n approach a specific, finite value.

If the value of r is 1, the series is in the form $a + a + a + \ldots$, which diverges.

If the value of r is -1, the series is in the form $a - a + a - a + \ldots$ which creates a sequence of partial sums in the form $S_1, S_2, S_3, \ldots = a, 0, a, 0, \ldots$ which while remaining finite, does not get closer and closer to a specific value, so in this case, the series diverges as well.

If the value of r is between -1 and 1 (exclusive,) however, the value of r^n will get closer and closer to zero as the value of n increases. Thus, in this case, and only this case, will the series converge, and it will converge to (because of the fact that r^n approaches zero):

$$\boxed{S = \frac{a}{1-r}}.$$

<u>The Harmonic Series</u>

One interesting series which comes up frequently in mathematics is called the harmonic series. Although it can be multiplied by a constant, the basic harmonic series is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

What makes the harmonic series interesting is that, despite the fact that the individual terms of the series approach zero, the series <u>diverges</u>.

To prove this, we will consider terms of the series in groups:

$$1 \geq \frac{1}{2}$$
$$\frac{1}{2} \geq \frac{1}{2}$$
$$\frac{1}{3} + \frac{1}{4} \geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Likewise, the next eight terms are all greater than or equal to $\frac{1}{16}$, and so their sum is greater than or equal to $\frac{1}{2}$. Adding together our inequalities, we find:

$$S_1 \geq \frac{1}{2}$$
$$S_2 \geq \frac{2}{2}$$
$$S_4 \geq \frac{3}{2}$$
$$S_8 \geq \frac{4}{2}$$
$$\ldots$$
$$S_{2^n} \geq \frac{n+1}{2}$$

Therefore, the sum of the first n terms of the harmonic series can be made as large as desired simply by choosing a sufficiently large value of n, and so does not converge.

Means

In the context of sequences, **means** are missing terms that come between terms of a sequence. Assume that we know the values of a_1 and a_3 for a sequence and wish to find the value of a_2. For this, we use a simple mean.

In an arithmetic sequence, we can find the mean as follows:

$$a_1 + a_3 = a + (a + 2d) = 2a + 2d$$
$$\frac{a_1 + a_3}{2} = a + d = a_2$$

Thus, the value $\boxed{\frac{a_1 + a_3}{2}}$ is defined to be the **arithmetic mean** of a_1 and a_3. Of course, the arithmetic mean is better known as a simple average.

The **geometric mean** of two numbers a_1 and a_3 is found similarly:

$$a_1 a_3 = a\left(ar^2\right) = a^2 r^2$$
$$\pm\sqrt{a_1 a_3} = ar = a_2$$

Note that the geometric mean may be either positive or negative, but in most contexts where it occurs, the positive value is usually the desired value, giving us the customary value $\boxed{\sqrt{a_1 a_3}}$ as the "formula" for the geometric mean of two numbers.

The **harmonic mean** of two numbers a_1 and a_3 can be found by recognizing that the *reciprocals* of the terms of a harmonic sequence form an arithmetic sequence, and so we can find the harmonic mean of two numbers by (a) flipping the numbers over, (b) finding their arithmetic mean, and then (c) flipping the result back over. That is:

$$\frac{1}{\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_3}\right)} = \frac{1}{\frac{a_1+a_3}{2a_1a_3}} = \frac{2a_1a_3}{a_1+a_3}$$

The harmonic mean $\frac{2a_1a_3}{a_1+a_3}$ is used most frequently in math team problems where you need to average speed over a constant distance as in this example:

"Sharon drove to work at an average speed of 60 mph and drove home along the same route at an average speed of 40 mph. What is her average speed for the day?"

One way to solve this problem would be to assume a travel distance of, say, 120 miles. This would result in a time of 2 hours to work and 3 hours coming home (long commute!) A total distance of 240 miles in 5 hours means an average speed of 48 mph (NOT 50 mph!)

We could also employ the harmonic mean here:

$$\frac{2(40)(60)}{40+60} = \frac{4800}{100} = 48.$$

<u>The Fibonacci Sequence</u>

Did you really think we could *not* bring this one up? The Fibonacci sequence, named after Leonardo of Pisa (a.k.a. Leonardo Fibonacci) is defined by

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2},\ n \ge 2$$

This leads to the famous sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$

Here are some interesting (and theoretically useful) facts about the Fibonacci sequence:

- The ratio of successive terms $\frac{a_{n+1}}{a_n}$ approaches the golden mean $\frac{\sqrt{5}+1}{2}$ as $n \to \infty$.

- The sum of the first n terms is given by $F_1 + F_2 + ... + F_n = F_{n+2} - 1$
- The 10th Fibonacci number (55) is also the 10th triangular number.
- The 12th Fibonacci number (144) is also the 12th square number.
- $F_{n-1}F_{n+1} = F_n^{\,2} + (-1)^n$

<u>One more thing: Arithmogeometric Series</u>

Actually, I have no idea if this is the right name for this type of series, but it does seem to come up from time to time on math contests, so that's what I'm calling it. Consider the following series:

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + ... + \frac{n}{2^n} + ...$$

This series has an arithmetic series in the numerators and a geometric series in the denominators. Let's take it as a given that such a series converges (it does) and attempt to find its sum.

We can start by setting the sum to S and the multiplying both sides by r:

$$\begin{aligned} S &= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + ... + \frac{n}{2^n} + ... \\ \frac{1}{2}S &= \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{64} + ... + \frac{n}{2^{n+1}} + ... \end{aligned}$$

Then, subtract the bottom equation from the top, pairing up terms with matching denominators:

$$\frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ...$$

Notice, though, that the right side is now a standard, convergent, geometric series, with $a = \frac{1}{2}$ and $r = \frac{1}{2}$. According to the formula above, the sum is $S = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$. Therefore,

$$\begin{aligned} \frac{1}{2}S &= 1 \\ S &= 2 \end{aligned}$$

Problems:

1. [NYSML 2006 I4] Note that 5 = 2 + 3 and 12 = 3 + 4 + 5. That is, 5 is the sum of 2 consecutive positive integers the smallest of which is 2, and 12 is the sum of 3 consecutive positive integers the smallest of which is 3. Compute the number of positive integers less than 1000 that, like 5 and 12, are the sum of n consecutive positive integers the smallest of which is that same n for some n > 1.

2. [NYSML 2005 I7] $x_0 = 4$, $x_1 = 9$, $x_2 = 200$, $x_3 = 5$, and $x_n = x_{n-1} - x_{n-2} + x_{n-3} - x_{n-4}$ for $n \geq 4$. Compute x_{2005}.

3. [ARML 2006 I5] Starting with $n = 1$, let $\{a_n\}$ be an infinite geometric sequence with $a_2 = 17$. Compute the smallest possible positive sum of the sequence.

4. [ARML 1999 I3] If a and b are the roots of $11x^2 - 4x - 2 = 0$, then compute the product:

$$(1 + a + a^2 + a^3 + ...)(1 + b + b^2 + b^3 + ...)$$

5. [NYSML 2001 I4] Which integer is closest to $\dfrac{1+3+5+...+497+499}{1+2+3+...+249+250}$?

6. [NYSML 2000 I6] Let $S = \sum_{n=1}^{\infty} \dfrac{\frac{1}{2}n(n+1)}{3^n}$. Compute S.