Practice #9 – Vector Basics

The topic of vectors is introduced to most high school students during physics classes. However, vector calculations can be used to perform geometric and trigonometric calculations, and are especially useful in 3-dimensional geometry.

Vector Basics

In mathematics, a **vector** is defined as an ordered n-tuple of numbers $\langle a_1, a_2, ..., a_n \rangle$. The value of n is called the **dimension** of the vector, thus $\langle x, y, z \rangle$ is a 3-dimensional vector. If a vector is placed on a set of coordinate axes, the vector is represented by a directed line segment going from a **tail** at the point with coordinates $(t_1, t_2, ..., t_n)$ to a **head** with coordinates $(h_1, h_2, ..., h_n)$. In this case, the coordinates of the vector are given by $\langle h_1 - t_1, h_2 - t_2, ..., h_n - t_n \rangle$. Note that the coordinates of the vector are only representative of the change in the coordinates between the two points, not the endpoints themselves. Thus, the vector that goes from the point $(1,2,3)$ to the point $(3,2,-5)$ is the vector $\langle 2,0,-8 \rangle$. This is the exact same vector as the one that goes from $(1,5,2)$ to $(3,5,-6)$, or the vector that goes from $(0,0,0)$ to $(2,0,-8)$. This is reflected in the fact that the a vector has only two defining properties, the **magnitude** (which is the distance between the head and the tail), and the **direction** of the vector. The position of the vector is not a property of the vector, and so it should be understood that two vectors with the same coordinates are to be considered the same vector, regardless of where the heads and tails of the vectors are located.

Notice that in the previous example, when the vector was placed with the tail at the origin, the head of the vector was located at the same coordinates as the vector itself. Such a vector is said to be in **standard position**, and if the vector is used to represent the location of a point in space, this vector is known as a **position vector**.

Finding Magnitude and Direction

The magnitude of any vector can be found by use of the n-dimensional distance formula:

$$\left\| \langle a_1, a_2, ..., a_n \rangle \right\| = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}$$

Problem: Find the shortest 3-dimensional vector $\langle x, y, z \rangle$ with all positive integer coordinates such that $x \le y \le z$ and which has a positive integer magnitude.

The direction of a vector is not a simple thing to describe unless the vector is 2-dimensional. In that case, we can use the fact that the angle of a line is related to its slope to produce the formula

$\tan\theta = \dfrac{y}{x}$ for the vector $\langle x, y \rangle$.

In more than two dimensions, direction is usually indicated by means of a **unit vector**, which is a vector of magnitude one in the same direction as the original vector. You can always find a unit direction vector simply by multiplying a vector by the reciprocal of its magnitude.

Problem: Find a unit vector in the same direction as your answer to the previous problem. Confirm that it is in fact a unit vector by using the magnitude formula.

Note that two vectors are **parallel**, that is, they have the same direction, if and only if one vector is a scalar multiple of the other, that is

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n} = k \,.$$

Standard Unit Vectors

Where useful, vectors can alternatively be expressed in terms of **standard unit vectors**. A standard unit vector is a vector consisting of all 0s with a 1 in only one place. Specifically, the vector u_k is a vector which has its kth coordinate equal to 1 and all other coordinates equal to zero. This allows us to express a vector as follows:

$$\langle a_1, a_2, \ldots, a_n \rangle = \sum_{k=1}^{n} a_k u_k$$

In three dimensions, the unit vectors are called

$$\hat{i} = \langle 1, 0, 0 \rangle$$
$$\hat{j} = \langle 0, 1, 0 \rangle$$
$$\hat{k} = \langle 0, 0, 1 \rangle$$

and so, for example we can write $\langle 3, -4, 8 \rangle$ as $3\hat{i} - 4\hat{j} + 8\hat{k}$. This notation is occasionally useful (see the section on cross product.)

Vector Operations

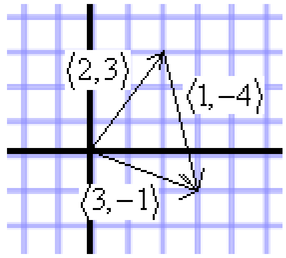
Vectors can be manipulated as well, as follows:

Vector addition: $\langle a_1, a_2, \ldots, a_n \rangle + \langle b_1, b_2, \ldots, b_n \rangle = \langle a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n \rangle$

Vector subtraction: $\langle a_1, a_2, \ldots, a_n \rangle - \langle b_1, b_2, \ldots, b_n \rangle = \langle a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n \rangle$

Multiplication by a scalar: $k \langle a_1, a_2, \ldots, a_n \rangle = \langle ka_1, ka_2, \ldots, ka_n \rangle$

When visualizing vectors as directed line segments, the concept of vector addition is most easily considered in terms of motions. If a vector of $\langle 2, 3 \rangle$ represents a movement 2 units to the right and 3 up, and a vector of $\langle 1, -4 \rangle$ represents a movement 1 to the right and 4 down, then the sum of those two vectors $(\langle 3, -1 \rangle)$represents the most direct path that would produce the same <u>result</u> as moving 2 to the right and 3 up, followed by moving 1 to the right and 4 down. Such a vector is called the **resultant**:



Scalar multiplication is much simpler: multiplying a vector by a scalar simply multiplies the magnitude of the vector by the absolute value of the scalar, keeping the direction of the vector the same if the scalar is positive, or reversing it if the scalar is negative.

Numerous mnemonics have been suggested for vector subtraction, but I can think of nothing simpler than just negating (i.e. reversing) the second vector and then adding.

$$\dot{a} - \dot{b} = \dot{a} + \left(-\dot{b}\right) = \dot{a} + (-1)\dot{b}$$

Note that vector addition and scalar multiplication obey the commutative, associative, and distributive properties (why?)

A vector can also be separated into its **components**, which are individual vectors parallel to the axes. For example, $\langle -2,9,4\rangle$ has x-component $\langle -2,0,0\rangle$, y-component $\langle 0,9,0\rangle$ and z-component $\langle 0,0,4\rangle$.

Unfortunately, there is no single method for multiplying two vectors, but instead there are two standard methods: the dot product and the cross product.

The **dot product** (or **inner product** or **scalar product**) of two vectors is defined as follows:

$$\vec{a} \cdot \vec{b} = \sum_{k=1}^{n} a_k b_k$$

Thus, if $a = \langle 1,2,3\rangle$ and $b = \langle 4,5,6\rangle$, then $a \cdot b = 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = 4 + 10 + 18 = 32$

It can be proven that (in any number of dimensions) that $\dot{a} \cdot \dot{b} = \|\dot{a}\|\|\dot{b}\|\cos\theta$, where θ is the (non-reflex) angle between the vectors. We can use this as a nice way of finding the angle between two vectors. Continuing the previous example, to find the angle between $\langle 1,2,3\rangle$ and $\langle 4,5,6\rangle$:

$$32 = \|\langle 1,2,3\rangle\|\|\langle 4,5,6\rangle\|\cos\theta$$
$$32 = \sqrt{14}\sqrt{77}\cos\theta$$
$$\theta = \cos^{-1}\left(\frac{32}{\sqrt{14}\sqrt{77}}\right) \approx \cos^{-1}(0.97463) \approx 12.933^\circ$$

The dot product is used for other purposes, including the physics concept of work and the statistical concept of correlation coefficient (r), but that's beyond the scope of this packet.

The **cross product** (or **vector product**) of two three-dimensional vectors is defined in terms of the determinant of a matrix. Since this packet is primarily concerned with vectors, we'll just spell out the formula here:

$$\langle x_1, y_1, z_1\rangle \times \langle x_2, y_2, z_2\rangle = \det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \hat{i}\,(y_1z_2 - z_1y_2) - \hat{j}\,(x_1z_2 - z_1x_2) + \hat{k}\,(x_1y_2 - y_1x_2)$$

Note a couple of things about the cross product:

- The terms on the right side of the formula each consist of a single standard unit vector times the determinant of its minor, which is the matrix you get when you remove the top row and the column containing that particular unit vector. Hence, the term containing the $\hat{i}$ does not contain any x terms, etc.
- Notice all the minus signs. As a consequence, cross product is not commutative.
- The result of a cross product is a vector, but the result of a dot product is a scalar.

Geometrically, the cross product of two vectors is always perpendicular to the plane containing them. What happens if the two vectors are parallel? In that case, there is no single plane containing the two vectors, but, no worries, the cross product of the two vectors will be zero when the vectors are parallel.

Problem: prove that last fact, that is, prove that $(a)\times(ka)=0$.

The magnitude of the cross product of two vectors is given by this (familiar-looking) formula:

$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin\theta$, where θ is the same angle used in the dot product formula.

Notice the similarity to the formula for the area of a triangle $K = \frac{1}{2}ab\sin C$. In fact, the cross product can be used to find the area of a parallelogram or triangle formed by two vectors by either finding the magnitude of the cross product (in the case of a parallelogram) or one-half the magnitude (in the case of a triangle.)

Problem: Find the area of the triangle whose vertices are (0,0,2), (0,4,0), and (6,0,0).

Complex Numbers as Vectors

It is somewhat well-known that complex numbers can be graphed on the coordinate plane using the correspondence

$$x + yi \Leftrightarrow \langle x, y\rangle$$

It makes sense to use a vector here since complex numbers add, subtract, and multiply by scalars in exactly the same fashion as vectors (prove it!) However, they multiply in an unusual way. Given a complex number $x+yi$, we can define r to be the magnitude of the vector, also called the **absolute value** of the complex number:

$$r = |x+yi| = \|\langle x, y\rangle\| = \sqrt{x^2+y^2}$$

Note that in this formula that's $\sqrt{x^2+y^2}$, <u>not</u> $\sqrt{x^2+(yi)^2}$. The second formula could produce an imaginary magnitude, which is impossible even here.

We then define θ to be the **argument** of the complex number, simply defined as the angle of the vector, using the usual directional compass for trigonometry (0° = right, 90° = up, etc.) The value of the argument can be found simply using:

$\tan\theta = \frac{y}{x}$ for the vector $\langle x, y\rangle$ or the complex number $x+yi$.

Once these values are known, we can express the coordinates of the vector using the standard circular function formulas $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$ to obtain

$$\langle x, y\rangle = \langle r\cos\theta, r\sin\theta\rangle$$

But, in complex numbers, this translates into

$$x + yi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta) = rcis\theta$$

where $cis\theta$ is defined as $cis\theta = \cos\theta + i\sin\theta$.

A complex number so written is said to be in **polar form**. For example, the number $1+\sqrt{3}i$ has magnitude 2 and argument 60°, and so $1+\sqrt{3}i = 2cis60^\circ$. While polar form is terrible for performing additions and subtractions (in these cases, complex numbers should be left in $x+yi$ form,) it is beautiful for multiplication, division, powers, and roots:

$$(r_1 cis\theta_1)(r_2 cis\theta_2) = (r_1 r_2)cis(\theta_1 + \theta_2)$$

$$(r_1 cis\theta_1)/(r_2 cis\theta_2) = (r_1/r_2)cis(\theta_1 - \theta_2)$$

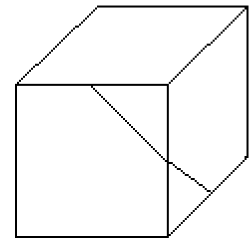
$$(rcis\theta)^n = r^n cis(n\theta)$$ [This is known as DeMoivre's Theorem.]

$$\sqrt[n]{rcis\theta} = \sqrt[n]{r}cis\left(\frac{\theta}{n} + \frac{360^\circ}{n}k\right), \text{ where } k = 0, 1, 2, \ldots, n-1$$

The last of these rules indicates a rather lovely fact—that the n^{th} roots of a complex number lie evenly spaced about a circle on the complex plane.

Problems:

1. A cube with edge length 2 is shown at right. Two line segments join the midpoints of three of the cube's edges. Find the angle formed by the two line segments.



2. [NYSML 2008 I7] $\frac{1+8i}{7+4i} = a+bi$, where a and b are real numbers and $i^2 = -1$. When the complex number $a+bi$ is plotted in the complex plane, compute its distance from the origin.

3. The vector $\langle 2,4,5 \rangle$ lies in standard position. Find the equation of the plane which passes through the point $(2,4,5)$ and is perpendicular to the vector $\langle 2,4,5 \rangle$. [Hint: what must be true about the dot product of perpendicular vectors?]

4. [NYSML 2002 I8] Given $O(0,0), P(4,3), Q(x,y)$ and $R(1,0)$ such that $PO = QO$ and $\angle QOP = 2\angle PQR$. Compute the ordered pair (x,y) if $y > 0$.

5. [AIME] Tetrahedron ABCD has vertices with coordinates $A(0,0,0)$, $B(2,0,0)$, $C(0,4,0)$, and $D(0,0,6)$. A sphere is inscribed within the tetrahedron. Find the radius of the sphere. [Note: despite the fact that this was an AIME problem, the answer may or may not be an integer.]