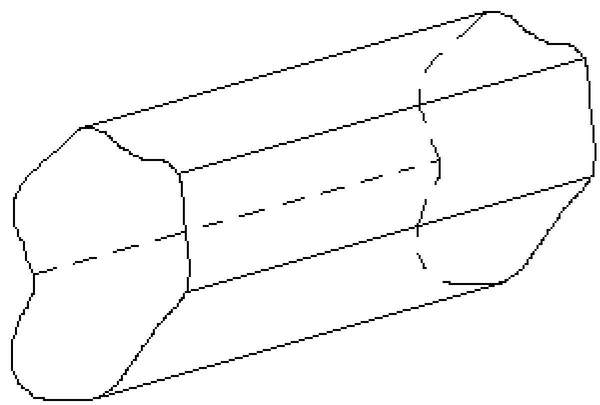
Practice #8 – Three-Dimensional (Solid) Geometry

Much like trigonometry, much of the material in solid geometry that we need is in the form of equations and formulas. There are a number of tricks and tips, however, and we'll try to summarize some of them here. Let's start with the simplest of three-dimensional shapes, the **prisms**.

Prisms

A prism is any solid consisting of two parallel **bases** which are congruent closed plane figures connected by a **lateral surface** which is perpendicular to the bases. The perpendicular distance between the bases is called the **height** of the prism. (I like to think of a prism as any shape that could be extruded through a Play-Doh Fun Factory™.) Examples of prisms include cubes, rectangular prisms, triangular prisms, cylinders, or just, in general, any shape of constant cross-section:



[In this paper, we will use P for perimeter, A for area, and V for volume.]

For all prisms, $V = A_{base}h$. This works for rectangular prisms $(V = lwh)$, triangular prisms $\left(V = \frac{1}{2}bah\right)$, and cylinders $(V = \pi r^2 h)$, as well as any other prismatic shape.

In addition, we can talk about the **lateral surface area** (LSA), the **base surface area** (BSA), and the **total surface area** (SA) of the prism:

$$LSA = P_{base}h \qquad BSA = 2A_{base} \qquad SA = P_{base}h + 2A_{base}$$

For rectangular prisms, we also have some specific useful relations:

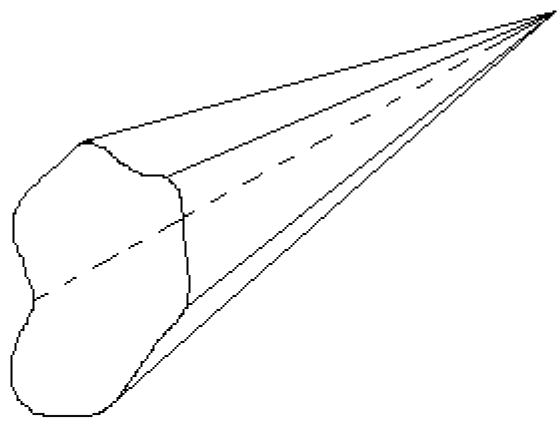
$SA = 2lw + 2wh + 2lh$ Total edge length = $TEL = 4l + 4w + 4h$

The length of the main diagonal of a rectangular prism is given by $l^2 + w^2 + h^2 = d^2$.

These lead to the following interesting relationship (prove it!): $(TEL/4)^2 = d^2 + SA$

Pyramids

If we replace the top base of a prism with a single point (the apex), which we then connect to every point of the base with a set of line segments, we obtain a solid called a **pyramid**. Examples include square and triangular pyramids, polygonal pyramids, cones, and general pyramids:



The volume of any pyramidal shape is given by $V = \frac{1}{3} A_{base} h$, where h is the distance from the apex to the base, measured perpendicularly.

Surface area formulas vary. In the case where the base of the pyramid is a polygon, the area of each lateral side generally has to be computed individually, though in the case of more symmetric pyramids, many of the lateral triangles will have the same area.

When the base is a circle and the apex is directly above the center of the circle, we have what is called a **right circular cone**. In this case, the lateral surface area is given by

$LSA = \frac{1}{2} Cl$, where C is the circumference of the base and $l = \sqrt{r^2 + h^2}$ is the **slant height**.

Curved Shapes

For a sphere, the volume and surface area are given by $V = \frac{4}{3}\pi r^3$ and $SA = 4\pi r^2$.

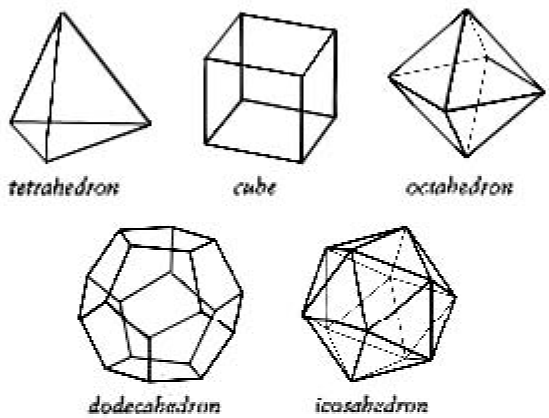
For an ellipsoid $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right)$, the volume is given by $V = \frac{4}{3}\pi abc$. Surface area is not simple.

The volume of a spherical cap of radius r and height h is given by $V = \frac{1}{3}\pi h^2 (3r - h)$.

The volume of a circular paraboloid with radius r and height h is given by $V = \frac{1}{2}\pi r^2 h$.

The Platonic Solids (Regular Polyhedra)

A **regular polyhedron** is defined to be a solid all of whose faces are congruent regular polygons and all of whose vertices are equivalent. There are only 5 regular polyhedra, known as the Platonic Solids (why only 5?):



Volumes for the first 3 can be generally easily derived. The last two, not so. If you really like formulas, enjoy!

$$V_4 = \frac{\sqrt{2}}{12}s^3 \quad V_6 = s^3 \quad V_8 = \frac{\sqrt{2}}{3}s^3 \quad V_{12} = \frac{15+7\sqrt{5}}{4}s^3 \quad V_{20} = \frac{15+5\sqrt{5}}{12}s^3$$

If you connect the center of each face of a polyhedron, using a line segment, to the center of each adjacent face, you obtain the **dual** of the polyhedron. When you do this, the number of edges remains constant, and the number of vertices and number of faces swap. Note that the cube and octahedron are duals of each other, as are the dodecahedron and icosahedron. The tetrahedron is its own dual.

One interesting invariant among all polyhedra is the **Euler Number**, which for polyhedra is 2. This is computed by using the formula $V + F - E = 2$, which relates the number of vertices, faces, and edges, and which holds for all polyhedra.

Three-Dimensional Analytic Geometry

This is much too large of a subject to go into fully here, so we'll just stick to a few basic concepts. If you construct a rectangular coordinate system with a third axis, called z, perpendicular to the both the x- and y- axes, you get the 3-D coordinate system. The x-, y-, and z- coordinates of a point represent the directed distances from the yz-, xz-, and xy- planes, respectively. Regardless of how you choose to orient the axes on paper, the positive directions must satisfy the **right-hand rule** for coordinate axes. The right hand rule states that if you point your right thumb in the positive direction of the x-axis and your right index finger in the positive direction of the y-axis, then your palm will be facing in the positive direction of the z-axis. For example, if you draw the x-axis to the right and the y-axis up, as normally illustrated, then, using the right-hand rule, you will find (try it!) that the positive direction of the z-axis will be towards you, out of the page.

Analogous Components of 3-D Analytic Geometry

Some of the formulas that you learned in 2 dimensions have 3 dimensional counterparts:

Item	2-D version	3-D counterpart
Midpoint (averaging 3 points finds the centroid of a triangle.)	$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$	$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$
Distance between 2 points	$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$	$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 + z_2)^2}$
Equation of a line (a plane, in 3-D)	$Ax + By + C = 0$	$Ax + By + Cz + D = 0$
Distance from a given point to a line (plane)	$\left\lvert \frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}} \right\rvert$	$\left\lvert \frac{Ax_0 + By_0 + Cz_0 + D}{\sqrt{A^2 + B^2 + C^2}} \right\rvert$

Problems:

1. [ARML 2006 T1] A cylinder of height 10cm and radius 6cm is turning at 1 revolution per minute. A paintbrush 4 cm wide starts at the top, with the top edge of the brush aligned with the top edge of the cylinder and slowly moves vertically down the cylinder at 1cm/minute. Compute the time in minutes it takes for the paintbrush to paint three-fifths of the lateral surface area of the cylinder.

2. [NYSML 2005 I6] A pyramid has a regular hexagon as a base and the other six faces are congruent isosceles triangles. The area of the base is 12 and the area of each of the triangular faces is 12. Compute the volume of the pyramid.

3. [NYSML 2005 I8] Let $\lfloor t \rfloor$ stand for the largest integer that is less than or equal to t. Let S be the solid

$$\left\{(x, y, z): 0 \leq x^2 + y^2 < 4,\ 0 \leq z \leq \left\lfloor x^2 + y^2 + 1 \right\rfloor\right\}.$$

Compute the volume of S.

4. [ARML 2000 I6] Let S be the sphere whose equation is $x^2 + y^2 + z^2 = 25$. Points $P(3,4,0)$ and $Q(3,-4,0)$ lie on S. Compute the number of planes containing P and Q that intersect S in a circle whose area is an integer.

5. [ARML 2003 T6] Let the faces of a unit cube be the six planes x = 0, y = 0, z = 0, x = 1, y = 1, and z = 1. Compute the values of t such that the points $A\left(a, \frac{1}{2}, t\right)$ and $B\left(\frac{1}{2}, 1, t\right)$ have <u>multiple</u> equal-length shortest paths connecting them along the faces of the unit cube.

6. [ARML 2003 T9] Assume that as a cubical bar of soap is used, all edges shrink at a constant rate of n units per day. Starting with a full bar, the soap was used for 6 days and its surface area was cut in half. Starting with a full bar of soap, compute exactly the time it would take for the volume to become one-eighth of the original volume.

7. [ARML 2001 T6] Compute the ratio of the volume of a sphere to the volume of the largest regular octahedron that will fit inside it.

8. [ARML 2001 T9]