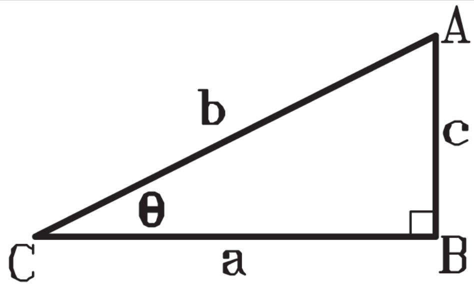
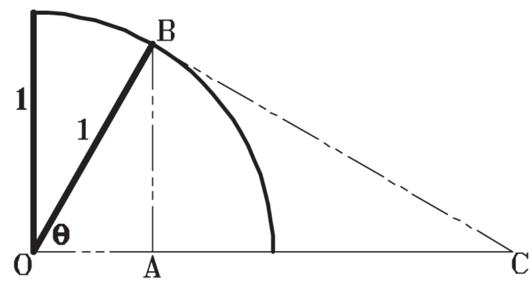
NASSAU COUNTY INTERSCHOLASTIC MATHEMATICS LEAGUE
ALL-STAR PROGRAM ∀*

Practice #7 – Trigonometry

There is not too much material used in math competitions beyond what is taught in standard high-school mathematics; however, formulas which were part of the reference tables for Regents mathematics are expected to be memorized for math competitions. With that in mind, I have reprinted the trigonometry formulas from the original formula packet here. We'll talk about which of these formulas are important to memorize for NYSML and ARML:



$\sin\theta = {}^{c}\!/_{b}$; $\cos\theta = {}^{a}\!/_{b}$; $\tan\theta = {}^{c}\!/_{a}$



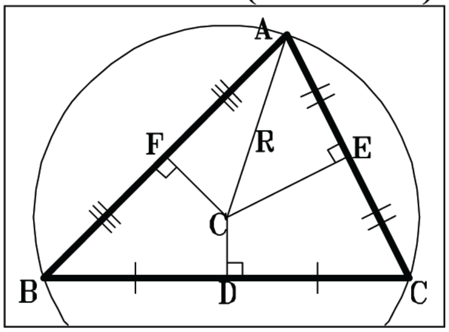
$\sin\theta = \overline{AB}$; $\cos\theta = \overline{OA}$; $\tan\theta = \overline{BC}$

θ	15°	18°	30°	36°	45°	54°	60°	75°
sin θ	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$	$\frac{\sqrt{2\left(5-\sqrt{5}\right)}}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$
cos θ	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{2\left(5+\sqrt{5}\right)}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2\left(5-\sqrt{5}\right)}}{4}$	$\frac{1}{2}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$
tan θ	$2-\sqrt{3}$	$\frac{\left(\sqrt{5}-1\right)\sqrt{2}}{2\sqrt{5+\sqrt{5}}}$	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{2\left(5-\sqrt{5}\right)}}{\sqrt{5}+1}$	1	$\frac{\left(\sqrt{5}+1\right)\sqrt{2}}{2\sqrt{5-\sqrt{5}}}$	$\sqrt{3}$	$2+\sqrt{3}$

Pythagorean	Odd-Even Functions	Summation of Angles
$\sin^2\theta + \cos^2\theta = 1$	$\sin(-\theta) = -\sin(\theta)$	$\sin(\theta \pm \gamma) = \sin(\theta)\cos(\gamma) \pm \cos(\theta)\sin(\gamma)$
$1 + \tan^2\theta = \sec^2\theta$	$\cos(-\theta) = \cos(\theta)$	$\cos(\theta \pm \gamma) = \cos(\theta)\cos(\gamma) \mp \sin(\theta)\sin(\gamma)$
$1 + \cot^2\theta = \csc^2\theta$	$\tan(-\theta) = -\tan(\theta)$	$\tan(\theta \pm \gamma) = \dfrac{\tan(\theta) \pm \tan(\gamma)}{1 \mp \tan(\theta)\tan(\gamma)}$

Multiple Angles			
	$\sin 2\theta = 2\sin\theta\cos\theta$	$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$	$\sin 4\theta = 4\cdot\sin\theta\cdot\cos\theta\cdot(\cos^3\theta - \sin\theta)$
	$\cos 2\theta = \cos^2\theta - \sin^2\theta$	$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$	$\cos 4\theta = \sin^4\theta + \cos^4\theta - 6\cos^2\theta\cdot\sin^2\theta$
	$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$	$\tan 3\theta = \dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$	$\tan 4\theta = \dfrac{4\tan\theta\left(1-\tan^2\theta\right)}{\tan^4\theta - 6\tan^2\theta + 1}$

Circumcenter ($\perp$bisectors)



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Extended Law of Sines

Area Formulas of the Triangle		
$K=\dfrac{ab\sin C}{2}$	$K=\dfrac{c^2\sin A\sin B}{2\sin C}$	$K=\dfrac{abc}{4R}$

Law of Cosines	$a^2 + b^2 = c^2 + 2ab \cos C$
Law of Tangents	$\tan(A)\tan(B)\tan(C) = \tan(A)+\tan(B)+\tan(C)$

Area Formulas of the Triangle					
$K=\dfrac{c \bullet h_c}{2}$	$K=\dfrac{ab\sin C}{2}$	$K=\dfrac{c^2\sin A\sin B}{2\sin C}$	$K=\dfrac{abc}{4R}$	$K=rs$	$K=\sqrt{s(s-a)(s-b)(s-c)}$
For planar triangle with vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3\ y_3)$					
$K = \dfrac{1}{2}\left\|\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right\|$			Coordinates of the centroid are $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$		

Sum to Product	Product to Sum
$\sin\theta \pm \sin\gamma = 2\sin\left(\dfrac{\theta\pm\gamma}{2}\right)\cos\left(\dfrac{\theta\mp\gamma}{2}\right)$	$\sin\theta \bullet \sin\gamma = \frac{1}{2}[\cos(\theta-\gamma) - \cos(\theta+\gamma)]$
$\cos\theta + \cos\gamma = 2\cos\left(\dfrac{\theta+\gamma}{2}\right)\cos\left(\dfrac{\theta-\gamma}{2}\right)$	$\cos\theta \bullet \cos\gamma = \frac{1}{2}[\cos(\theta-\gamma) + \cos(\theta+\gamma)]$
$\cos\theta - \cos\gamma = -2\sin\left(\dfrac{\theta+\gamma}{2}\right)\sin\left(\dfrac{\theta-\gamma}{2}\right)$	$\sin\theta \bullet \cos\gamma = \frac{1}{2}[\sin(\theta-\gamma) + \sin(\theta+\gamma)]$
$\tan\theta \pm \tan\gamma = \dfrac{\sin(\theta\pm\gamma)}{\cos\theta \bullet \cos\gamma}$	$\tan\theta \bullet \tan\gamma = \dfrac{\cos(\theta-\gamma)-\cos(\theta+\gamma)}{\cos(\theta-\gamma)+\cos(\theta+\gamma)}$

Square Identities	Cube Identities	½ Angle Identities	tan (θ/2) Identities
$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$	$\sin^3\theta = \dfrac{3\sin\theta - \sin 3\theta}{4}$	$\sin\theta/2 = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$	$\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$
$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$	$\cos^3\theta = \dfrac{3\cos\theta + \cos 3\theta}{4}$	$\cos\theta/2 = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$	$\cos\theta = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$
$\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$	$\tan^3\theta = \dfrac{3\sin\theta - \sin 3\theta}{3\cos\theta + \cos 3\theta}$	$\tan\theta/2 = \pm\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}$	

Trigonometry and Analytic Geometry

It can be proven (try it!) that given any straight line $y = mx + b$, the angle θ formed between that line and any horizontal line satisfies the equation

$$\tan\theta = m,$$

with the angle measured to the upper-right of the point of intersection. Use of the difference identity for tangents implies that the angle θ formed between two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2},$$

with the sign determined by whether you're measuring the acute or obtuse angle so formed.

Question: show that this formula is consistent with the fact that perpendicular lines have negative reciprocal slopes.

More Fun withTangent

One of the more interesting and useful parts of the tangent function is that it provides a one-to-one correspondence between the real numbers and a finite-measure subset of the real numbers, by using the domain $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. This can result in some interesting problems. Take this for example:

Show that, given any 7 distinct real numbers, at least one pair (x, y) of them will satisfy the inequality

$$\frac{1 + xy}{x - y} > \sqrt{3}.$$

This question becomes much easier if you consider both x and y to be the tangents of angles (A, B) in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then we get:

$$\frac{1 + \tan A \tan B}{\tan A - \tan B} > \sqrt{3}$$

$$0 < \frac{\tan A - \tan B}{1 + \tan A \tan B} < \frac{1}{\sqrt{3}}$$

$$0 < \tan(A - B) < \frac{1}{\sqrt{3}}$$

$$0^\circ < A - B < 30^\circ$$

The rest of the proof is easy by comparison.

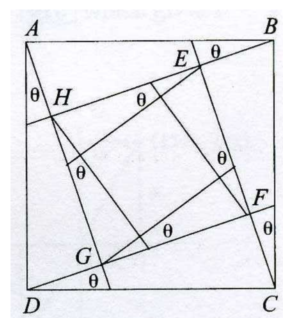
So let's get to the problems!

1. [NYSML 2008 I5] Compute $\sin(2008^\circ)\cos(28^\circ) - \sin(28^\circ)\cos(2008^\circ)$.

2. [NYSML 2003 I7] Compute the smallest $\theta > 0$ for which $\sec\theta = \log_{\sqrt{3}}\left(1/\log_{\sqrt[3]{\pi}}(\pi)\right)$. [$\theta$ is measured in radians.]

3. [NYSML 1999 I5] The altitudes of a parallelogram are of lengths 10 and 8, and they intersect at an angle whose sine is ¼. Compute the area of the parallelogram.

4. [ARML 2001 I5] For $m > 0$, $f(x) = mx$. The acute angle θ formed by the graphs of f and f^{-1} is such that $\tan\theta = \frac{5}{12}$. Compute the sum of the slopes of f and f^{-1}.



5. [ARML 2001 I6] In unit square $ABCD = S_0$, two pairs of parallel lines are drawn, each making an angle of θ with a side of S_0. These lines produce an inner square $EFGH = S_1$. The procedure is repeated using S_1 and angle θ, producing an inner square S_2. The process is continued. If $\theta = 15^\circ$, compute thes um of the areas of squares $S_0, S_1, S_2, S_3, \ldots$

6. [ARML 2000 I7] The measure of the vertex angle of isosceles triangle ABC is θ and the sides of the triangle are $\sin\theta, \sqrt{\sin\theta},$ and $\sqrt{\sin\theta}$. Compute the area of ΔABC.

7. [NYSML 1998 I6] If $\sec x + \cos x = 3$, compute the value of $\frac{\sec^{10} x + 1}{\sec^5 x}$.

8. [ARML 1999 I4] The measure of $\angle PAQ$ is 60°, AB bisects $\angle PAQ$ and circles P and Q are tangent to AB. If the radii of circles P and Q are 1 and 2 respectively, compute the distance from P to Q.