$\forall^*$

Practice #5: Advanced Algebra

Like the last practice, this will, out of necessity, be a bit of a random sampling of algebraic techniques and tricks that you will, hopefully, find useful. So let's start with the problems straight away:

Arithmetic Through Algebra

1. Compute x where $2^8 + 2^8 + 2^8 + 2^8 = 8^x$. [NYSML 2008 I1]

This problem is a classic example of "arithmetic through algebra," a technique wherein a seemingly complex arithmetic problem is made easier by using algebraic methods on the numbers involved. Consider the following relay-type problem:

- Compute the product of 1991 and 2009 as quickly as you can.

This would seem like a difficult task and would require a certain amount of hand computation; however, if you recognize that $1991 \cdot 2009 = (2000 - 9)(2000 + 9) = 2000^2 - 9^2 = 4{,}000{,}000 - 81 = 3{,}999{,}919$ the problem becomes a lot easier. Now try this one:

2. Compute $\sqrt{1600 - 240 + 9}$.

Completing the Square (and cube!)

Since any trinomial of the form $x^2 + 2ax + a^2$ can be factored into $(x + a)^2$, it follows that given any binomial of the form $x^2 + bx$, we can force factorization into the square of a binomial by adding the quantity $\left(\frac{b}{2}\right)^2$, thus giving us $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$. Of course, algebraically, if we add that term in, we will have to compensate in some way, usually by adding a quantity to the other side of the equation or by simultaneously adding and subtracting the quantity from a given expression.

For example, $x^2 + 28x - 288$ can be factored by completing the square:

$$\begin{aligned} &x^2 + 28x + 14^2 - 14^2 - 288 = (x + 14)^2 - 484 = (x + 14)^2 - 22^2 \\ &= (x + 14 + 22)(x + 14 - 22) = (x + 36)(x - 8) \end{aligned}$$

Of course, completing the square can be used for other purposes, such as solving quadratic equations, locating the centers/vertices of conic sections, and putting equations into standard form, but as this is an advanced algebra packet, we will leave it at that.

Completing the cube is a bit rarer and a bit trickier. Since $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$, we can use completing the cube any time the coefficients of x^2 and x are in the form $3a$ and $3a^2$ respectively.

For example, we could factor $x^3 + 9x^2 + 27x + 26$ as follows:

$$x^3 + 9x^2 + 27x + 27 - 27 + 26 = (x+3)^3 - 1$$
$$= (x+3-1)\left[(x+3)^2 + (x+3) + 1\right]$$
$$= (x+2)(x^2 + 7x + 13)$$

Here are a couple of practice problems for you to try:

3. For some $a > 0$, the graph of $y = x^3 - x^2 + ax$ intersects the x-axis in exactly two places. Compute a. [NYSML 2005 I4]

4. The graph of $y = x^3 + 6x^2 + 12x + 18$ is symmetric about the point (a,b). Compute the ordered pair (a,b). [NYSML 2002 I3]

<u>More Factoring Tricks</u>

In addition to the usual types of factoring taught in Regents mathematics, you should be familiar with the factorization of the difference of like powers:

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$$

and the sum of <u>odd</u> powers:

$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots - ab^{n-2} + b^{n-1})$, with signs alternating throughout.

5. The sum of even powers cannot be factored in this fashion over the real numbers. Can they be factored over the complex numbers in this way? If so, how? If not, why not?

6. The quantity $\dfrac{1}{\sqrt[3]{3} - 1}$ can be expressed in the form $\dfrac{\sqrt[3]{a} + \sqrt[3]{b} + c}{d}$ with positive integer a, b, c, d. Compute the smallest possible value of $a + b + c + d$.

In addition to the standard "factoring by grouping" there is another sort of factorization that can come in handy on occasion for expressions like $xy + 2x - 5y$. Can you come up with a good name for it?

$$xy + 2x - 5y = xy + 2x - 5y - (2)(5) + 10 = (x-5)(y+2) + 10$$

In this technique, given a trinomial of the form $xy + ax + by$, the quantity ab can be added to the end and the resulting four-term polynomial (quadrinomial?) can be factored by grouping. As with completing the square, the added term will generally need to be compensated for somehow.

Try these:

7. Find all lattice points that lie on the graph of $xy+2x+3y=9$.

Dividing Polynomials

Polynomials can be divided in the same way that integers can be divided, yielding a quotient and remainder, in the same manner as integers, to wit:

Division Theorem for Integers: For any pair of integers a and b, $b>0$, there exists a unique quotient q and remainder r, both integers, such that $a=bq+r,\ 0\le r<b$.

Division Theorem for Polynomials: For any pair of polynomials $a(x)$ and $b(x)$, $\deg(b)>0$, there exists a unique quotient $q(x)$ and remainder $r(x)$ such that $a(x)=b(x)q(x)+r(x)$, $0\le\deg(r)<\deg(b)$.

If the remainder polynomial $r(x)=0$, then we say that b is a **factor** of a.

For your information (or review), here are a couple of theorems regarding polynomial division:

Remainder Theorem: If $p(x)$ is divided by $(x-r)$, the remainder is $p(r)$.

Factor Theorem: $(x-r)$ is a factor of $p(x)$ if and only if $p(r)=0$.

Once it is known that $(x-r)$ is a factor of $p(x)$, the quotient can be found by synthetic division or long division. These are basic algebraic techniques which will not be reviewed here.

8. When the polynomial $P(x)$ is divided by $x-1776$, the remainder is 1776. When $P(x)$ is divided by $x+1776$, the remainder is -1776. Compute the remainder when $P(x)$ is divided by x^2-1776^2.

Sum and Product of Roots of Polynomial Equations

It is a well-known high school algebra fact that, given a quadratic equation $ax^2+bx+c=0$, the sum of the roots (solutions) of the equation is given by $-\frac{b}{a}$ and the product of the roots by $\frac{c}{a}$. These formulas can be simplified a bit by noticing that any polynomial can be made into a **monic polynomial** (a polynomial with leading coefficient 1) by dividing by the leading coefficient. Thus, the equation $ax^2+bx+c=0$ can be changed into $x^2+\frac{b}{a}x+\frac{c}{a}=0$. We can then rename the coefficients to get the following:

In any quadratic equation $x^2+Ax+B=0$, the sum of the roots is $-A$ and the product is B.

This result can be generalized to higher degree equations, and is one of the more useful facts to know for advanced mathematics competitions. For example, take the cubic case:

Let p,q,r be the roots of the equation $x^3 + Ax^2 + Bx + C = 0$. Then:

$-A = p + q + r$ (the sum of the roots taken one at a time)

$B = pq + pr + qr$ (the sum of the roots taken two at a time, in all possible combinations.)

$-C = pqr$ (the sum of the roots taken three at a time, i.e. the product of the roots.

The pattern continues into higher degrees, with the signs given alternating. So, for example, if the roots of a fourth-degree equation are given by p,q,r,s, then the coefficient of x^2 represents the sum of the roots taken two at a time: $B = pq + pr + ps + qr + qs + rs$.

<u>Sum/Product Expression</u>

As a final note, it can be proven that any sum of powers of two variables $x^n + y^n$ can be expressed as a combination of the sum $S = x + y$ and product $P = xy$ of the two variables. Here are the first few formulations:

$$x^2 + y^2 = (x+y)^2 - 2xy = S^2 - 2P$$
$$x^3 + y^3 = (x+y)^3 - 3xy(x+y) = S^3 - 3PS$$
$$x^4 + y^4 = (x+y)^4 - 4xy\left(x^2+y^2\right) - 6x^2y^2 = S^4 - 4P\left(S^2 - 2P\right) - 6P^2 = S^4 - 4PS^2 + 2P^2$$

9. Determine the formulation for $x^5 + y^5$.

10. $x^7 + y^7 = ?$ (Multiple choice)

(A) $S^7 - 7PS^5 + 14P^2S^2 - 7P^3S$ (B) $S^7 - 7PS^5 + 14PS^2 - 7P^2S$ (C) $S^7 - 7PS^5 + 14P^2S^3 - 7P^3S$

(D) $S^7 - 7PS^4 + 14PS^3 - 7P^3S$ (E) $S^7 - 7PS^4 + 14PS^2 - 7P^3S$

11. Determine a rule that, given the expressions for $x + y, x^2 + y^2, ..., x^n + y^n$, can be used to quickly determine the expression for $x^{n+1} + y^{n+1}$