NASSAU COUNTY INTERSCHOLASTIC MATHEMATICS LEAGUE
ALL-STAR PROGRAM

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Practice #4: Advanced Geometry

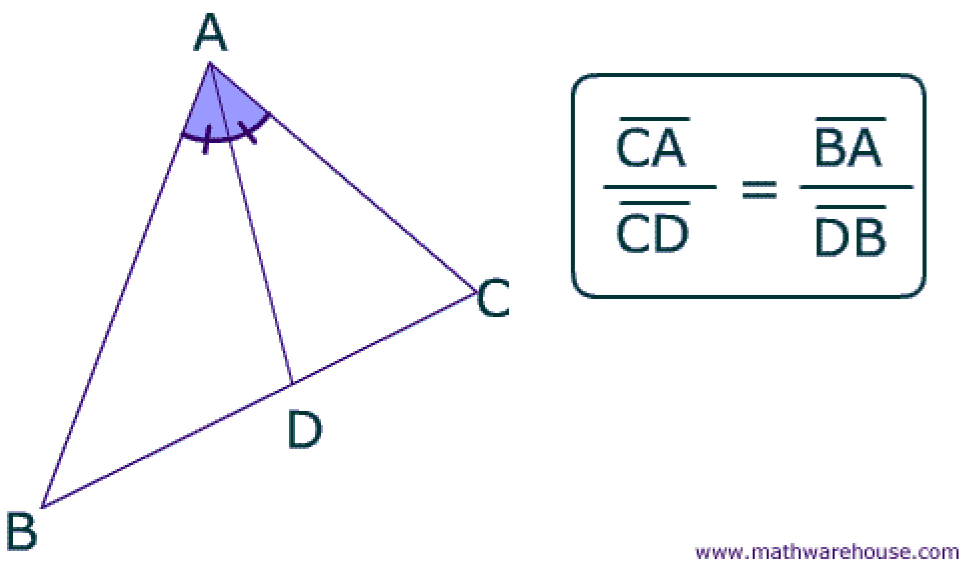
Since this is a (necessarily brief) summary of advanced geometry, we’re going to dispense with basic definitions and postulates and just move on to look at some problems and the ideas/theorems required (or just useful) to solve them.

<u>Stuff about Triangles</u>

1. In ΔABC, let D be a point on $\overline{BC}$ such that $\overline{AD}$ bisects $\angle A$. If AD = 6, BD = 4, and DC = 3, then find AB.

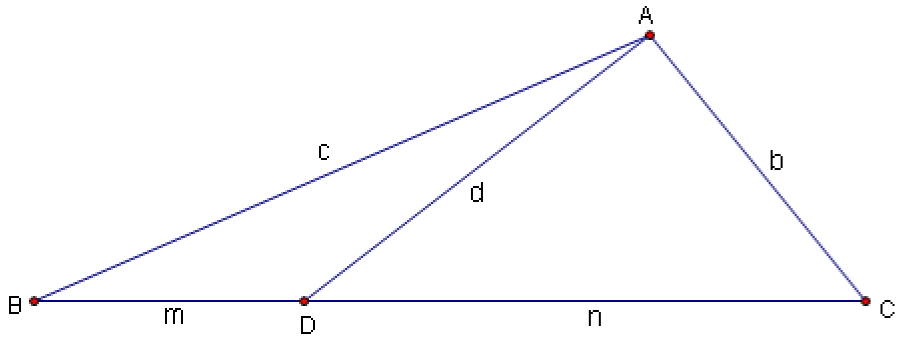
Although there are numerous ways to solve this problem, one particularly efficient method involves a combination of the Angle Bisector Theorem and Stewart’s Theorem.

The **Angle Bisector Theorem** states that an angle bisector in a triangle will cut the opposite side into segments which are in proportion to their adjacent triangle sides:



(For the reader: prove the Angle Bisector Theorem. How many ways can you find?)

Stewart's Theorem is an equation which provides a relationship between the sides of a triangle, a **cevian** (a line segment connecting one vertex to any point on the opposite side) and the segments into which the cevian divides the opposite side:



In this diagram, note that A is drawn at the top and B and C are lettered from left to right, alphabetically. The sides a, b, and c are lettered in accordance with usual convention. The cevian is given length d, and the segments of side BC are lettered m and n from left to right, alphabetically.

By using this set of letters, we get a useful mnemonic:

$$man + d^2a = b^2m + c^2n$$
$$man + dad = bmb + cnc$$

"A <u>man</u> and his <u>dad</u> made a <u>bomb</u> in the <u>sink</u>."

(Not exactly politically correct, but still useful.)

Now, using these two ideas, revisit problem 1.

Sometimes, there are other ideas that can come in handy, including adding lines to a diagram. Let's start with this prelude:

2. Use Stewart's Theorem (or the law of cosines) to prove that in a parallelogram, the sum of the squares of the diagonals equals the sum of the squares of the sides.

Then, tackle this one:

3. Find the length of median $\overline{AD}$ of ΔABC if AB = 5, BC = 7, and AC = 8.

<u>Stuff about Cyclic Quadrilaterals</u>

A **cyclic polygon** is any polygon that can be inscribed in a circle, that is, it is possible to find a single circle such that all the vertices of the polygon lie on the circle. Cyclic quadrilaterals have some useful properties. From basic circle geometry, it is relatively easy to show that, given cyclic quadrilateral ABCD:

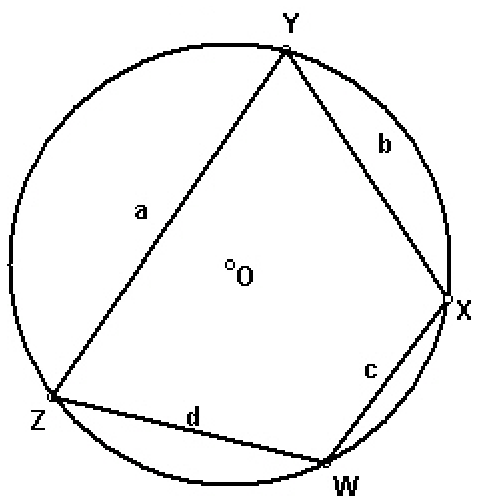
4. Prove that $m\angle A + m\angle C = m\angle B + m\angle D = 180^\circ$.

5. Prove that $\angle ABD \cong \angle ACD$ and state three other pairs of analogously congruent angles.

What is interesting and useful about the previous two statements is that the theorems implied ("If ABCD is a cyclic quadrilateral, then $m\angle A + m\angle C = 180^\circ$.") are, in fact, biconditionals ("ABCD is a cyclic quadrilateral if and only if $m\angle A + m\angle C = 180^\circ$.") Therefore, the conditions stated in problems 4 and 5 can be used to prove that a given quadrilateral is cyclic. (For the record, proofs of the converses of problems 4 and 5 work by drawing a circle through three of the four points (how do we know we can always draw a circle through 3 noncollinear points?) and showing that the fourth point can neither lie inside nor outside the circle, thus showing that the four points are **concyclic** (lying on the same circle.))

Here are a couple of formulas based on cyclic quadrilaterals:

Brahmagupta's Formula:



$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where K stands for area of the quadrilateral and s is the **semiperimeter**, equal to one-half of the perimeter.

(How does Brahmagupta's Formula relate to Hero(n)'s Formula?)

Ptolemy's Theorem:

If we name the diagonals in the previous diagram e and f, then $ac + bd = ef$.

6. Find the diagonal length of an isosceles trapezoid with bases of lengths 8 and 20 and legs of length 10.

7. Inscribed in a circle is a quadrilateral having sides of lengths 25, 39, 52, and 60 taken consecutively. What is the diameter of the circle?

Stuff about Areas of Triangles

Here, all in one place are a whole mess of formulas for the area of a triangle:

$$K = \frac{1}{2}bh = \frac{1}{2}ab\sin C = sr = \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)}$$

Legend:
h = altitude drawn to side AC
s = semiperimeter
r = inradius = radius of the inscribed circle
R = circumradius = radius of the circumscribed circle.

8. Prove $K = sr$. The simplest way is to break the triangle up into 3 smaller triangles.

9. Prove $K = \frac{1}{2}ab\sin C$.

10. Prove that, for any triangle, $\frac{a}{\sin A} = 2R$. Since 2R is the diameter of the circumscribed circle and since any vertex could be labeled A, this proves the **extended law of sines**: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

11. Use the results of questions 9 and 10 to prove $K = \frac{abc}{4R}$.

12. A triangle has sides of length 3, 5, and 7. Compute the circumradius of the triangle.

Problems:

1. In ΔABC, $\angle C = 90^\circ$, AB = 41, and AC = 40. Compute the radius of the circle inscribed in the triangle. [NYSML 2008 T2]

2. Point P is inside rectangle ABCD with PA = 15, PB = 6, and PC = 10. Compute PD. [NYSML 2008 T7] (By the way: what if point P is outside rectangle ABCD?)

3. Two circles of radius r intersect at two points, A and B. The line segment connecting the centers of the circles intersects the two circles at points C an D. Triangles ACD and BCD are equilateral with sides of length s. Compute s/r. [NYSML 2007 I7]

4. A pyramid has a regular hexagon as a base and the other six faces are congruent isosceles triangles. The area of the base is 12 and the area of each of the triangular faces is 12. Compute the volume of the pyramid. [NYSML 2005 I6]

5. Let ABCD be a trapezoid with E on segment $\overline{BC}$, F on segment $\overline{AD}$, and $\overline{AB} \parallel \overline{EF} \parallel \overline{CD}$. Exactly half of the area of ABCD is on either side of $\overline{EF}$. The lengths of the segments $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ are, respectively, p, q, and m; and p, q, and m are all prime numbers with 1 < p < m < q. Compute the smallest possible value for p. [NYSML 2006 I6]