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Practice #3: Probability and Combinatorics

Part 1: Probability

The term **probability** refers to the likelihood of a particular **event** occurring on a given random **trial**. Probability is measured in terms of what fraction of the time the event occurs (or will occur) over a large number of trials. Since an event can't occur less than zero times or more times than the number of trials, the probability of an event must lie between zero and one, inclusive.

There are two types of probability:

- **Observed probability** is the fraction of the time that the event actually occurred, given a certain number of trials:
  - $P_{obs}(E) = \dfrac{\text{number of times E occured}}{\text{number of trials}}$
- **Theoretical probability** is the predicted fraction of the time that an event will occur, given what is known about the situation:
  - $P(E) = \dfrac{n(E)}{n(S)}$, where:
    - $n(E)$ is the number of outcomes that satisfy the event, and
    - $n(S)$ represents the number of outcomes in the **sample space**, which is the set of all possible (equally-likely) outcomes. If the outcomes are not equally likely, then alternate methods must be employed.

As an example, consider the probability of getting a prime number on a roll of one die. Here, $E = \{2,3,5\}$ and $S = \{1,2,3,4,5,6\}$. Thus $P(prime) = \frac{3}{6} = \frac{1}{2}$. Of course, observed probability will be different; if we roll the die 20 times, we might get a prime number only 8 times, leading to an observed probability $P_{obs} = \frac{8}{20} = \frac{2}{5}$. Even if the observed probability is exactly equal to the theoretical probability, one more trial will automatically render these two quantities unequal. However, the **law of large numbers** states that as the number of trials approaches infinity, the two probabilities must converge:

$\lim_{n \to \infty} P_{obs}(E) = P(E)$, where *n* is the number of trials.

One way of determining the theoretical probability of an event, according to the law of large numbers, is the **Monte Carlo** method. The Monte Carlo method consists of simply performing a large number of trials and computing the observed probability, frequently by using a computer simulation. By using a sufficiently large number of trials, the observed probability will give a pretty good idea of the theoretical probability.

If each outcome of a trial has an associated value, then the **expected value** of a trial is the average value, per trial, that you should expect to get over a large number of trials. The formula for expected value is:

$EV = P_1V_1 + P_2V_2 + ... + P_nV_n$, summed over all possible outcomes 1, 2, …, n

where P_i represents the probability of outcome i and V_i represents the value of outcome i.

1. In a game, a fair die is rolled and a player receives 2^n points for a roll of *n*. What is the expected value for a roll of the die?

Given two separate events A and B, we can consider the probability that either both events occur or neither event occurs:

- AND formula: $P(A \wedge B) = P(A)P(B \mid A)$
- OR formula: $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

In the AND formula, $P(B \mid A)$ represents the probability of "B given A", which is the probability that B occurs, given that A has already occurred. The AND formula can also be used to compute **conditional probability**. By rearranging terms in the first formula, we can see:

$$P(B \mid A) = \frac{P(A \wedge B)}{P(A)}$$

2. A hat contains three cards: one is black on both sides, the second is white on both sides, and the third is black on one side, white on the other. You reach into the hat and pull out a card and place it on the table without looking at it, seeing that the top of the card is black. What is the probability that the underside of the card is also black?

Solution: Let A = "the top side is black" and B = "the underside is black". Since 3 of the 6 card sides are black, $P(A) = \frac{1}{2}$. Since 1 of the 3 cards is black on both sides, $P(A \wedge B) = \frac{1}{3}$. From the formula for conditional probability, $P(B \mid A) = \frac{1/3}{1/2} = \frac{2}{3}$.

If two events are **independent**, the probability that each event occurs remains the same regardless of the outcome of the other event. In that case, $P(B \mid A) = P(B)$ and so the AND formula becomes $P(A \wedge B) = P(A)P(B)$.

The OR formula is a case of the **inclusion-exclusion principle**. This principle can be used to find the probability that one or more events has occurred, the number of elements in the union of two or more sets, or the area of a shape consisting of two or more overlapping shape. The basic concept behind the inclusion-exclusion principle is that if we were to simply add together the probabilities of the two events, the outcomes common to both events would be counted twice, and so in order to correct this over-count, the overlapping outcomes are subtracted once.

To see how the inclusion-exclusion principle applies to counting elements of sets, consider the sets $A = \{1,2,3,4,5\}$ and $B = \{1,3,5,7,9\}$. If we want to compute the number of elements in the set $A \cup B$ by adding the sizes of the two sets, we get 10. However, we've counted the elements of set $A \cap B$ twice. By subtracting the 3 elements in the intersection, we find that the number of elements in set $A \cup B$ is 5 + 5 – 3 = 7. Symbolically, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

The inclusion-exclusion principle can be extended to 3 events as follows:

$$P(A \vee B \vee C) = P(A) + P(B) + P(C) - P(A \wedge B) - P(A \wedge C) - P(B \wedge C) + P(A \wedge B \wedge C).$$

To extend it to 4 or more events, simply alternate adding events taken 1 at a time, subtracting events taken 2 at a time, adding events taken 3 at a time, etc…

<u>Part 2: Combinatorics</u>

Part of the problem of computing probability often consists of determining the number of outcomes that satisfy a given event, or the number of possible outcomes in general. Sometimes, counting the number of outcomes individually can be difficult or impractical. For this purpose, we can use **combinatorics**, the science of counting without actually counting.

The most important combinatorial formula is the **counting principle**, which states that given a selection of sets, the number of combinations that can be formed by taking one item from each set is equal to the product of the sizes of the sets.

As an example, consider the number of outcomes of rolling two dice. Since each die has 6 separate outcomes, there are 36 possible rolls obtainable. If we wanted to determine, say, the probability of rolling a sum of 5, note that we could roll a 1, 2, 3, or 4 on the first die (with each case allowing only one possibility for the second die) meaning that there are 4 possible outcomes satisfying the event of rolling a sum of 5, yielding a probability of 4/36 or 1/9.

The counting principle leads immediately to the **permutation** formula: the number of <u>ordered</u> sequences of r items obtainable from a set of n items is given by first choosing one item from the set (n choices), then choosing one from the remainder (n-1 choices), the one from what's left (n-2 choices,) continuing until r items are chosen. This is written:

$${}_nP_r = n(n-1)(n-2)\ldots(n-(r-1)) = n(n-1)(n-2)\ldots(n-r+1)$$

We can simplify this formula by multiplying by $\frac{(n-r)!}{(n-r!)}$, yielding

$${}_nP_r = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

It is important to note that this formula considers ordering when counting possibilities; when counting 3-letter permutations from the string ABCDE, the permutations ACD, CDA, and ADC would all be counted separately in ${}_5P_3 = 5 \cdot 4 \cdot 3 = 60$.

Sometimes, however, when choosing elements from a set, the order that the items are chosen does not matter—all that matters is which elements are chosen. In this case, we are counting **combinations**. To find a formula for the number of combinations, we can start with the permutation formula. However, if we count permutations, note that for every possible combination of r items, we have counted each of the ${}_rP_r = r!$ orderings of those r items separately. Therefore, the permutation formula yields a number which is equal to exactly $r!$ times too large. So,

$${}_nC_r = \binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

The two notations on the left both indicate the number of combinations of r items chosen from a set of n items.

3. Prove (a) ${}_nC_1 = n$, (b) ${}_nC_{n-1} = n$, (c) ${}_nC_{n-r} = {}_nC_r$, (d) ${}_nC_n = 1$, (e) ${}_nC_0 = 1$, and

(f) ${}_nC_0 + {}_nC_1 + \ldots + {}_nC_n = 2^n$

If not all of the items in a set are unique, the formula for the number of orderings for that set is different. The number of **permutations with repetition** of a set of n items is given by $\frac{n!}{r_1!r_2!\ldots r_k!}$, where r_i represents the number of repetitions of item i. For example, the number of permutations of the letters of the word MISSISSIPPI is given by $\frac{11!}{4!4!2!1!}$. (Proof?)

Sometimes, the best way to count a set of combinations or permutations is to establish a one-to-one correspondence between your set and another set which we know how to count. Consider this problem:

4. How many ways are there to arrange a set of 6 black balls and 10 white balls in a row such that nowhere are there two black balls next to each other?

This is ostensibly a permutation problem, however, if we consider placing the 10 white balls in a row first:

_ O _ O _ O _ O _ O _ O _ O _ O _ O _ O _

we can create a legal row by placing the 6 black balls in 6 of the 11 spaces between and outside the 10 white balls. Since it doesn't matter in which order we pick the 6 spaces, this is actually a combination problem, and the answer is ${}_{11}C_6 = \frac{11!}{6!5!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6! \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 11 \cdot 1 \cdot 3 \cdot 2 \cdot 7 = 462$.

<u>Part 3: Geometric Probability</u>

As a rule, when a problem asks you for the probability that a randomly-chosen point falls within a certain region, the problem comes down to a ratio of areas. Take this example:

5. A dartboard consists of two concentric circles with radii 3 and 5, respectively. A point is chosen uniformly at random within the outer circle. What is the probability that the point falls within the inner circle?

Answer: $P = \frac{\pi(3)^2}{\pi(5)^2} = \frac{9}{25}$.

However, there is a significant warning to be understood here: in order for this area ratio to work, the point must be chosen randomly. If we were to consider the previous problem using polar coordinates, by choosing two numbers randomly in the intervals $0 \le r \le 5$ and $0 \le \theta < 2\pi$, then the probability of the point (r,θ) falling within the inner circle is 3/5.

Try this one yourself:

6. A point E is chosen uniformly at random from within square ABCD. Find $P(m\angle AEB > 90^\circ)$.

Warm-up problems:

1. The ordered pair of positive integers $(x, y) = (1000, 1)$ is one of many solutions to the equation $2x + 8y = 2008$. Among the 81 ordered pairs of positive digits (a, b), how many of the equations $ax + by = 2008$ have solutions (x, y) which are ordered pairs of positive integers? [NYSML 2008 T3]

2. Compute the number of elements of the set $S = \{1, 2, 3, ..., 2008\}$ that, like 10 and 2008 for example, have 0 as at least one of their digits. [NYSML 2008 T6]

3. Define a *strictly decreasing number* as a number whose digits have the property that each digit is greater than the digit immediately to its right—that is, 7421 is strictly decreasing but 682 and 9773 are not. Compute the number of 5-digit strictly decreasing numbers.

4. Define a *decreasing number* as a number whose digits have the property that each digit is greater than or equal to the digit immediately to its right—that is, 7421 and 9773 are decreasing but 682 is not. Compute the number of 5-digit decreasing numbers.

Problems:

1. Call the four-digit positive integer *abcd* "Triple Outer Inner", or "TrOI" for short, if *a, b, c, d* are digits such that $a \geq 1;\ b, c, d \geq 0;\ d = 3 \times a;\ \text{and}\ c = 3 \times b$. For example, 1263 and 2006 are TrOI integers. Compute the number of four-digit TrOI positive integers. [NYSML 2006 I1]

2. The sum of the squares of the digits of 2003 is $2^2 + 0^2 + 0^2 + 3^2 = 13$. For how many four-digit positive integers is the sum of the squares of the digits 13? [NYSML 2003 I4]

3. There are $45^2 = 2025$ pairs (B, C) of two-digit odd positive integers. For how many of these pairs (B, C) does the quadratic equation $x^2 + Bx + C = 0$ have a pair of integers as solutions? [NYSML 2003 I7]

4. A digit is chosen at random from the set of odd digits. A second integer is chosen at random, also from all the odd digits. Compute the probability that the sum of the two digits is a multiple of 3. [NYSML 2003 I2]

5. Two players, Billy and Trilly, are playing a game by taking turns rolling a fair six-sided die. If Billy rolls a multiple of 2, he wins. If Trilly rolls a multiple of 3, she wins. Thrilly goes first. What is the probability that Thrilly wins the game?

6. Dr. Bunsen Honeydew drops a glass rod on the floor which breaks in two uniformly randomly-chosen points on the rod. What is the probability that the three pieces can be rearranged to form a triangle?