Nassau County Interscholastic Mathematics League

Team Contest Answers must be integers from 0 to 999, inclusive. 2025 – 2026

Calculators are allowed.

Time: 40 minutes

31. Compute xy if $2^x = 2^{\sqrt{2}} + 2^{\sqrt{2}}$ and $2^y = \frac{2^{\sqrt{2}}}{2}$.

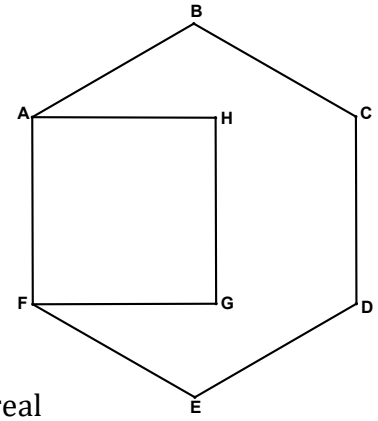
32. A Lucas Number is a number in an infinite series of numbers, where the first two numbers are given, and the next number is the sum of the preceding two numbers. For example, 1, 5, 6, 11, 17, 28… . In a given set of Lucas Numbers, the 18th number is 2026 and the 21st number is 7912. Compute the 17th number in the same set.

33. In $\Delta ABC, m\measuredangle A = 75°, m\measuredangle B = 60°$, and $AB = 24$. If $BC = x + y\sqrt{z}$ in simplest radical form, compute xyz.

34. The coordinates of the vertices of ΔABC are $A(-15,2), B(-2,10)$, and $C(5,0)$. Compute the sum of the squares of the coordinates of the centroid of ΔABC. Note that the centroid of a triangle is the point of intersection of its three medians.

35. Compute the minimum value of the domain of the real-valued function: $g(x) = \sqrt{1 - \log_5(35 - 5x)}$.

36. In the accompanying figure, square$AFGH$ and regular hexagon $ABCDEF$ have side $\overline{AF}$ in common. If the area of regular hexagon $ABCDEF$ is $486\sqrt{3}$, compute the area of square$AFGH$.



37. Compute $c + \frac{27}{a}$ if $a + \frac{1}{b} = 3$ and $b + \frac{3}{c} = \frac{1}{3}$ where a, b, and c are real numbers.

38. If $f(2x - 1) + f(3) = 4x - 1$ and $f(2) = \frac{p}{q}$ in simplest form, compute pq.

39. If $0 \le x \le 2\pi$, then the sum of the roots of the equation $\sin x + \cos x = \frac{\sqrt{2}}{2}$, expressed in simplest form, is $\frac{a}{b}\pi$. Compute ab.

40. Compute the number of positive integers that are relatively prime to 720 and that are also less than 720.

Solutions for Team Contest

31. From $2^x = 2^{\sqrt{2}} + 2^{\sqrt{2}} \rightarrow 2^x = 2 \cdot 2^{\sqrt{2}} = 2^{\sqrt{2}+1} \rightarrow x = \sqrt{2} + 1$. From $2^y = \frac{2^{\sqrt{2}}}{2} = 2^{\sqrt{2}-1}$ $\rightarrow y = \sqrt{2} - 1 \rightarrow xy = \left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right) = \mathbf{1}$.

32. Let $a = 17^{\text{th}}$ number, $b = 2026 = 18^{\text{th}}$ number, $c = 19^{\text{th}}$ number, $d = 20^{\text{th}}$ number, and $e = 7912 = 21^{\text{st}}$ number. So, $a + b = c,\ b + c = d,$ and $c + d = e$. From the second and third equations, $c = d - b$ and $c = e - d$. Adding these equations gives $2c = e - b \rightarrow 2c = 7912 - 2026 \rightarrow c = 2943$. Thus, from the first equation, $a + 2026 = 2943 \rightarrow a = \mathbf{917}$. Alternatively, if x is the 17^{th} term, then the terms are: $x,\ 2026,\ 2026 + x,\ 4052 + x,\ 6078 + 2x$. Thus, $6078 + 2x = 7912 \rightarrow x = 917$.

33. From vertex A, draw altitude $\overline{AD}$. Then ΔABD is a 30°-60°-90° triangle and ΔADC is an isosceles right triangle. Therefore, $BD = 12$ and $AD = DC = 12\sqrt{3}$. Then $BC = 12 + 12\sqrt{3}$ and the required product is **432.**

34. The coordinates of the centroid of a triangle are the averages of the x and y-coordinates of the vertices of the triangle. This can be proved using the fact that the centroid of a triangle is located $2/3$ of the way along each median from any vertex to the midpoint of the opposite side. So, the x-coordinate of the centroid is $\frac{-15+(-2)+5}{3} = -4$ and the y-coordinate is $\frac{2+10+0}{3} = 4$. Thus, $(-4)^2 + 4^2 = \mathbf{32}$.

35. In order for the function to be real-valued, $1 - \log_5(35 - 5x) \geq 0 \rightarrow$ $\log_5(35 - 5x) \leq 1 \rightarrow 35 - 5x \leq 5 \rightarrow 5x \geq 30 \rightarrow x \geq 6$. Also, since the argument of a log is positive, $35 - 5x > 0 \rightarrow 5x < 35 \rightarrow x < 7$. So, the minimum value of the domain of this function is **6.**

36. Let $x = AF$. The area of regular hexagon $ABCDEF$ is six times the area of the equilateral central triangle of the regular hexagon. Then, $6 \cdot \frac{x^2\sqrt{3}}{4} = 486\sqrt{3} \rightarrow$ $x^2 = 81 \cdot 4 \rightarrow x^2 = 324$. The required area is **324.**

37. Solve the given equations for b: $b = \frac{1}{3-a}$ and $b = \frac{1}{3} - \frac{3}{c} \rightarrow \frac{1}{3-a} = \frac{1}{3} - \frac{3}{c} \rightarrow \frac{1}{3-a} = \frac{c-9}{3c} \rightarrow$ $\frac{3c}{c-9} = 3 - a \rightarrow a = 3 - \frac{3c}{c-9} = \frac{3c-27-3c}{c-9} \rightarrow a = \frac{-27}{c-9} \rightarrow c + \frac{27}{a} = c + \frac{27}{\frac{-27}{c-9}} = c - (c-9) = \mathbf{9}$.

38. Set $2x - 1 = 3 \rightarrow x = 2 \rightarrow 2f(3) = 7 \rightarrow f(3) = \frac{7}{2} \rightarrow f(2x-1) = 4x - \frac{9}{2}$.

    Then set $2x - 1 = 2 \rightarrow x = \frac{3}{2} \rightarrow f(2) = 4 \cdot \frac{3}{2} - \frac{9}{2} = \frac{3}{2} \rightarrow pq = \mathbf{6}$.

39. Square both sides to yield $(\sin x)^2 + (\cos x)^2 + 2\ \sin x \cos x = \frac{1}{2} \rightarrow 1 + \sin 2x = \frac{1}{2} \rightarrow \sin 2x = -\frac{1}{2} \rightarrow 2x = \frac{7\pi}{6} + 2k\pi$ or $\frac{11\pi}{6} + 2k\pi \rightarrow 2x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6} \rightarrow x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$. The roots, $\frac{11\pi}{12}$ and $\frac{19\pi}{12}$ are extraneous, so the sum of the roots is $\frac{5\pi}{2}$ and the required product is **10**.

40. Two positive integers are relatively prime if their greatest common divisor is 1. The prime factorization of 720 is $2^4 \cdot 3^2 \cdot 5$. So, by Euler's totient function, the number of integers less than 720 that are relatively prime to 720 is $720\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = 720\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right) =$ **192**.

    Alternatively, use a Venn diagram: The overlap of all 3 circles contains $720 \div 30 = 24$ numbers with 30 as a factor. The overlaps between pairs of circles contain $720 \div 6 = 120$, $720 \div 10 = 72$, and $720 \div 15 = 48$ numbers, respectively. (Note that 24 is already part of each region). Finally, $720 \div 2 = 360$, $720 \div 3 = 240$, and $720 \div 5 = 144$, which are the totals in each circle. Therefore, $96 + 48 + 24 + 24 = 192$.

