Nassau County Interscholastic Mathematics League

Team Contest Answers must be integers from 0 to 999, inclusive. 2024 – 2025

Calculators are allowed.

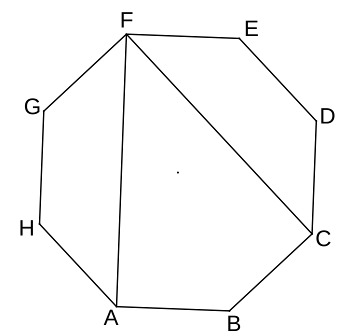
Time: 40 minutes

31. If, for all values of x, $(x+5)(x+a) = x^2 + bx + 15$, compute b.

32. A harmonic sequence is defined as a sequence whose reciprocals are an arithmetic sequence. If $2, x, y, 5$ is a harmonic sequence and $x + y = \frac{p}{q}$ in simplest form, compute $p + q$.

33. Each edge of a cube measures 4 feet long. Three faces of the cube are $ABCD$, $CDEF$, and $BCFG$. In simplest form, if the area of ΔBDF is $p\sqrt{q}$ square feet, compute $p + q$.

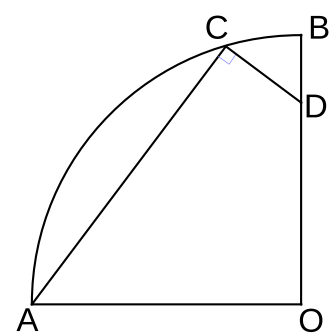
34. Compute the number of perfect square divisors of 256,000,000.



35. In the figure, regular octagon $ABCDEFGH$ has a perimeter of 16. Diagonals $\overline{AF}$ and $\overline{CF}$ trisect $\measuredangle GFE$. If the area of quadrilateral $ABCF$ can be expressed in simplest form as $a + b\sqrt{c}$, compute $a + b + c$.

36. The sum of the solutions of $4(\sin x)^2 - 3 = 0$ on the interval $[0, 4\pi)$ is $k\pi$. Compute k.

37. If $\log\left(\frac{x^3}{y^4}\right) = -1$, $\log \sqrt[8]{x^4y^2} = 1$, and, in simplest form, $\log(xy) = \frac{p}{q}$, compute pq.



38. Given a quarter circle with a center at point O. Point C is on minor arc $\overset{\frown}{AB}$ and point D is on $\overline{OB}$ such that $AC = 24$, $DC = 7$, and $m\measuredangle ACD = 90°$. Compute OD.

39. The lengths of the altitudes of a triangle are 6, 8, and h. If $h < x$, compute the minimum possible value of x.

40. If $a^3 + 4a = 8$, compute $a^7 + 64a^2$.

Solutions for Team Contest

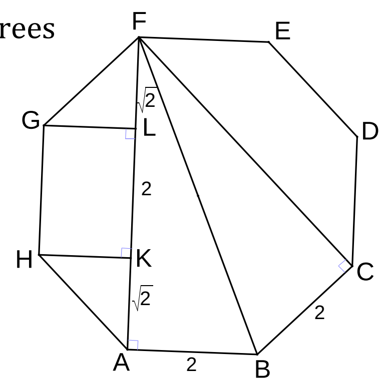
31. Since $(x+5)(x+a) = x^2 + 5x + ax + 5a = x^2 + bx + 15$, then
$5a = 15 \rightarrow a = 3 \rightarrow 5 + 3 = b \rightarrow b = \mathbf{8}$.

32. From the given condition, $\frac{1}{2}, \frac{1}{x}, \frac{1}{y}, \frac{1}{5}$ is an arithmetic sequence. Using $a_n = a_1 + d(n-1)$, the common difference, $d = \frac{\frac{1}{5}-\frac{1}{2}}{3} = -\frac{1}{10}$. Therefore, the arithmetic sequence is 0.5, 0.4, 0.3, 0.2.
Then $x + y = \frac{5}{2} + \frac{10}{3} = \frac{35}{6}$. The required sum is **41.**

33. Each of the sides of ΔBDF is a diagonal of a face of the cube. Therefore, ΔBDF is equilateral whose side-length is $4\sqrt{2}$ feet. The area of an equilateral triangle is determined by
$K = \frac{s^2\sqrt{3}}{4} = \frac{(4\sqrt{2})^2\sqrt{3}}{4} = 8\sqrt{3}$ and the required sum is **11.**

34. The given number equals $2^{14} \cdot 5^6$. Each divisor of 256,000,000 has the form $2^a 5^b$. For any divisor to be a perfect square, both a and b must be even. Therefore, there are 8 choices for a (0, 2, 4, 6, 8, 10, 12, 14) and 4 choices for b (0, 2, 4, 6). Thus, there are **32** perfect square divisors of 256,000,000.

35. The measure of each interior angle of the regular polygon is 135 degrees and $\measuredangle FAB$ and $\measuredangle FCB$ are right angles. From vertices H and G, drop perpendiculars to diagonal $\overline{AF}$. They intersect diagonal $\overline{AF}$ at points K and L respectively. Thus, triangle AKH and triangle FLG are isosceles right triangles with each hypotenuse of length 2 and $AK = FL = \sqrt{2} \rightarrow CF = AF = 2 + 2\sqrt{2}$. Then, the areas of triangle ABF and triangle CBF are each $\frac{1}{2} \cdot 2 \cdot (2 + 2\sqrt{2}) = 2 + 2\sqrt{2}$, Thus, the area of quadrilateral $ABCF$ is $4 + 4\sqrt{2}$ and the required sum is **10.**



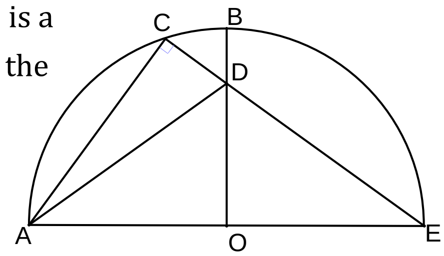
36. From the given, $(\sin x)^2 = \frac{3}{4} \rightarrow \sin x = \pm\frac{\sqrt{3}}{2} \rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}$.

The sum of the 8 solutions on the given interval is 16π. Thus, $k = \mathbf{16}$.
[There are many other dissections that could be used to arrive at the same result.]

37. Note that $\log\left(\frac{x^3}{y^4}\right) = -1 \rightarrow 3\log x - 4\log y = -1$ and $\log \sqrt[8]{x^4y^2} = 1 \rightarrow \frac{1}{2}\log x + \frac{1}{4}\log y = 1 \rightarrow 2\log x + \log y = 4 \rightarrow 8\log x + 4\log y = 16 \rightarrow 11\log x = 15 \rightarrow \log x = \frac{15}{11} \rightarrow 3 \cdot \frac{15}{11} - 4\log y = -1 \rightarrow 4\log y = \frac{45}{11} + 1 \rightarrow \log y = \frac{14}{11}$.

So, $\log(xy) = \log x + \log y = \frac{15}{11} + \frac{14}{11} = \frac{29}{11}$. The required product is **319**.

38. Complete the semi-circle and make diameter $\overline{AOE}$. Since $< ACE$ is a right angle, $\overrightarrow{CD}$ intersects $\overline{AOE}$at point E. Draw $\overline{AD}$. Then, from the Pythagorean Theorem in $\Delta ACD, AD = 25$. Since $\Delta ADO \cong \Delta EDO, DE = 25$. From the Pythagorean Theorem in $\Delta ACE, AE = 40$ and $AO = OE = 20$. Finally, from the Pythagorean Theorem in $\Delta ADO, OD = \mathbf{15}$.



39. The altitudes of lengths 6, 8, and h are drawn to the sides with lengths a, b,and c respectively. If we represent the area of ΔABC as k, then $k = \frac{6a}{2} = \frac{8b}{2} = \frac{ch}{2} \rightarrow a = \frac{k}{3},\ b = \frac{k}{4}$, and $c = \frac{2k}{h}$. From the triangle inequality, $\frac{k}{3} < \frac{2k}{h} + \frac{k}{4}$ and $\frac{2k}{h} < \frac{k}{3} + \frac{k}{4} \rightarrow \frac{k}{3} - \frac{k}{4} < \frac{2k}{h} < \frac{k}{3} + \frac{k}{4} \rightarrow \frac{1}{12} < \frac{2}{h} < \frac{7}{12} \rightarrow \frac{1}{24} < \frac{1}{h} < \frac{7}{24} \rightarrow \frac{24}{7} < h < 24$. The minimum value of x is **24.**

40. From the given,

$a^3 + 4a = 8$. Multiply each term by $16a$ to get

$16a^4 + 64a^2 = 128a$

$64a^2 = 128a - 16a^4$ (1)

$a^3 = 8 - 4a$ (2) Now square both sides to get

$a^6 = 64 - 64a + 16a^2$ (3)

Then, the required sum

$$
\begin{aligned}
a^7 + 64a^2 &= a^7 + 128a - 16a^4 \quad \text{from (1)} \\
&= a^7 - 16a^4 + 128a \\
&= a(a^6 - 16a^3 + 128) \\
&= a(a^6 - 16(8 - 4a) + 128) \quad \text{from (2)} \\
&= a(64 - 64a + 16a^2 - 128 + 64a + 128) \quad \text{from (3)} \\
&= a(16a^2 + 64) \\
&= 16(a^3 + 4a) \\
&= 16 \times 8 \quad \text{from given} \\
&=128
\end{aligned}
$$

Alternative Solution: Use the calculator to solve $a^3 + 4a - 8 = 0 \rightarrow a = 1.364 \ldots$

Then, $a^7 + 64a^2 = \mathbf{128}$.