Nassau County Interscholastic Mathematics League

Contest #2 Answers must be integers from 0 to 999, inclusive. 2024 – 2025

Calculators are allowed.

Time: 10 minutes **Name: ______________________________**

7. The mean of five numbers is 66. The median of these numbers is 65. Their mode is 71 and their range is 11. Compute the value of the second smallest number.

8. Compute x, if $3^{3x} + 9^{3x/2} + 27^{x} = 3^{2026}$.

7. [ ]

8. [ ]

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9. Compute the remainder when 5^{2025} is divided by 100.

10. The probability, expressed in simplest form, that the letters in the word MATHEMATICS are randomly arranged and as a result spell out the word MATHEMATICS is $\frac{p}{q}$. Compute $\frac{q}{64800}$.

9. [ ]

10. [ ]

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11. Compute one less than the smallest positive integer that is both a cube and a multiple of 50.

12. If $8^x + \frac{1}{8^x} = 9$, compute $2^{9x} + \frac{1}{2^{9x}}$.

11. ☐ 12. ☐

Solutions for Contest #2

7. Call the numbers a_1, a_2, a_3, a_4, a_5, with $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$. Note that $a_3 = 65 \rightarrow a_4 = a_5 = 71 \rightarrow a_1 = 60$. The sum of the numbers is the product of 5 and 66, or 330. Then $a_2 = 330 - (60 + 65 + 2 \cdot 71) = \mathbf{63}$.

   Alternate Solution: Since there are 5 numbers with a mean of 65 and a mode of 71, the numbers must be __, __, 65, 71, 71. The least number must be 60 since the range is 11. Finally, 71 is 5 more than the mean, 65 is 1 less than the mean, and 60 is 6 less than the mean. The sum of the numbers less than the mean must equal the sum of the numbers above the mean so $6 + 1 + x = 5 + 5$, so $x = 3$. The second least number is 3 less than the mean or 63.

8. Convert each expression into base 27: $3^{3x} = 27^x$, and $9^{\frac{3x}{2}} = 27^x$. It follows that, $27^x + 27^x + 27^x = 3^{2026} \rightarrow 3 \cdot 27^x = 3^{2026} \rightarrow 3 \cdot 3^{3x} = 3^{2026} \rightarrow 3^{3x+1} = 3^{2026} \rightarrow$ $3x + 1 = 2026 \rightarrow 3x = 2025 \rightarrow x = \mathbf{675}$.

9. Every integral power of 5 where the power is greater than 1 has 25 as its final two digits. So, the remainder when divided by 100 is **25**.

10. Because MATHEMATICS has three letters which occur twice each, the number of permutations of MATHEMATICS is $\frac{11!}{2!\cdot 2!\cdot 2!}$. Therefore, the required probability is $1/\frac{11!}{2!\cdot 2!\cdot 2!} = \frac{8}{11!} = \frac{1}{4{,}989{,}600.}$. Therefore, $q = 4{,}989{,}600$ and $\frac{4{,}989{,}600}{64{,}800} = \mathbf{77}$.

11. Note that $50 = 2^1 \cdot 5^2$. The exponents of a cube's prime factors must be multiples of 3. The number that is needed is $2^3 \cdot 5^3 - 1 = \mathbf{999}$.

12. From the given, $2^{3x} + \frac{1}{2^{3x}} = 9$. Let $y = 2^{3x}$. Find $y^3 + \frac{1}{y^3} = \left(y + \frac{1}{y}\right)\left(y^2 + \frac{1}{y^2} - 1\right)$.
    Since $y + \frac{1}{y} = 9$, $y^2 + \frac{1}{y^2} + 2 = 81 \rightarrow y^2 + \frac{1}{y^2} = 79$.
    Thus, $2^{9x} + \frac{1}{2^{9x}} = y^3 + \frac{1}{y^3} = (9)(79 - 1) = \mathbf{702}$.