Nassau County Interscholastic Mathematics League

Team Contest   Answers must be integers from 0 to 999, inclusive.   2022 – 2023

Calculators are allowed.

Time: 40 minutes

31. The coordinates of the endpoints of $\overline{AB}$ are $A(x, 8)$ and $B(-4, -4)$. The coordinates of point M, the midpoint of $\overline{AB}$, are $(10,2)$. Compute x.

32. Compute the smallest positive integral multiple of 12 which leaves a remainder of 3 when divided by 51.

33. Compute the maximum value of x so that 2^x divides 23!

34. Two different numbers are selected randomly from {1, 2, 3, 4, 5, 6} without replacement. The probability that their product is greater than 15, in simplest form, is $\frac{p}{q}$. Compute $p + q$.

35. In $\Delta XYZ, m\measuredangle X = 60$ degrees, $m\measuredangle Y = 70$ degrees, and $XY = 4$. Point W lies on $\overline{XZ}$ with $XW = 2$. Compute the degree measure of the smallest angle of ΔYWZ.

36. In a rectangular coordinate system, there are two circles that each contain the point whose coordinates are (4,3) and that are also tangent to both coordinate axes. Compute the sum of the radii of the two circles.

37. The expression $\left|\sum_{k=1}^{1982} kx^k \sin\left(\frac{k\pi}{2}\right)\right|$ is a polynomial. Find the sum of the coefficients of the odd powered terms.

38. The lengths of the bases of a trapezoid are 10 and 40. The degree-measures of the base angles on the longer base are 30 and 60. Compute the distance between the midpoints of the two bases.

39. Compute $72 \cdot \left(\frac{1}{x} + \frac{1}{y}\right)$ if $25^x = 4^y$ and $5^{4x} = 10^6$.

40. Compute the sum of all solutions of $(x^2 - 18x + 31)^{(x^2-6x+8)} = 1$.

Solutions for Team Contest

31. The x-coordinate of the midpoint of a segment is the average of the abscissas of the endpoints. Therefore, $\frac{x-4}{2} = 10 \ \rightarrow x = \mathbf{24}$.

32. We are looking for integers x and y for which $12x = 51y + 3 \rightarrow 4x = 17y + 1$. We need a multiple of 17 that is 1 less than a multiple of 4. The smallest is 51. Therefore, $y = 3$ and $x = 13$ and $12x = \mathbf{156}$.

33. Divisors of 23! that are multiples of 2 include 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, and 22. These have 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, and 1 factors of 2, respectively. Thus, there are **19** factors of 2 in 23!.

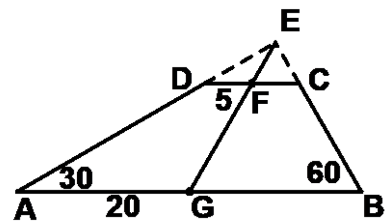
34. The only successful products greater than 15 are (6)(5), (6)(4), (6)(3), and (5)(4). The number of possible selections is $\binom{6}{2} = \frac{6\cdot 5}{2} = 15$. Therefore, the probability that their product is greater than 15 is $\frac{4}{\binom{6}{2}} = \frac{4}{15}$ and the required sum is **19.**

35. First, $m\measuredangle Z = 50$ degrees. Note that in $\Delta XWY, \frac{XW}{XY} = \frac{1}{2}$ and $m\measuredangle X = 60$ degrees. Then, use the Law of Cosines in ΔXWY to see that $YW = 2\sqrt{3}$ and ΔXWY is a 30-60-90 triangle with $m\measuredangle XYW = 30$ degrees. Thus, the smallest angle in ΔYWZ is $\measuredangle ZYW$ whose measure in degrees is **40**.

36. In order for the circles to be tangent to each of the two axes, the center must lie on the line whose equation is: $y = x$. Therefore, if the radius of one of the circles is r, then the center of the circle has coordinates (r, r). So, $(r-4)^2 + (r-3)^2 = r^2 \rightarrow$ $r^2 - 14r + 25 = 0$. The sum of the roots of the equation is the sum of the two radii which is **14.**

37. The given polynomial can be written as $|1 \cdot 1x + 2 \cdot 0x^2 + 3 \cdot -1x^3 + 4 \cdot 0x^4 + 5 \cdot 1x^5 + 6 \cdot 0x^6 + 7 \cdot -1x^7 + 8 \cdot 0x^8 + \cdots + 1981 \cdot 1x^{1981} + 1982 \cdot 0x^{1982}|$. The coefficients of the even powered terms are all zero and the coefficients of the odd powered terms are: 1, –3, 5, –7, 9, –11, ..., 1977, – 1979, 1981. Group and add as follows: (1 – 3) + (5 – 7) + (9 – 11) + ... + (1977 – 1979) + 1981. There are 495 groups of –2 which equals –990 plus 1981 for a total of 991.
Alternatively, the given series can be separated into two arithmetic series:
$S_1 = 1 + 5 + 9 + \cdots + 1981$ and $S_2 = -3 - 7 - 11 - \cdots - 1979$. Using the formula $a_n = a_1 + d(n-1)$, S_1 has 496 terms and S_2 has 495 terms. So, using the formula $S = \frac{n}{2}(a_1 + a_n)$, $S_1 = 491{,}536$ and $S_2 = -490{,}545$. Thus, $S_1 + S_2 = \mathbf{991}$.

38. In the diagram, extend the rays containing the legs $\overrightarrow{AD}$ and $\overrightarrow{BC}$ and let them intersect at point E. Note that $m\measuredangle AEB = 90°$, creating two right triangles, each with median $\overline{EFG}$. Its length is half the hypotenuse. So, $DF = 5 = FC = EF$ and $AG = GB = EG = 20$. Thus, $FG = 20 - 5 = \mathbf{15}$.



39. The given equations result in $5^{2x} = 1000$ and $4^y = 1000 \rightarrow x = \frac{3}{2\log 5}$ and $y = \frac{3}{2\log 2}$. So,

$$72 \cdot \left(\frac{1}{x} + \frac{1}{y}\right) = 72 \cdot \left(\frac{2\log 5}{3} + \frac{2\log 2}{3}\right) = 72 \cdot \left(\frac{\log 25 + \log 4}{3}\right) = 72 \cdot \frac{\log 100}{3} = 72 \cdot \frac{2}{3} = 48.$$

Alternatively, $25^x = 4^y \rightarrow 5^{2x} = 2^{2y}$. Also, $5^{4x} = 10^6 \rightarrow (5^{2x})^2 = 10^6 \rightarrow 2^{2y} \cdot 5^{2x} = 10^6$. This leads to $2^{2y} \cdot 5^{2x} = 2^6 \cdot 5^6 \rightarrow x = 3$ and $y = 3$.
Thus, $72\left(\frac{1}{3} + \frac{1}{3}\right) = 72\left(\frac{2}{3}\right) = \mathbf{48}$.

40. Case I: If $x^2 - 6x + 8 = 0$ and $x^2 - 18x + 31 \neq 0$, then the solution set is $\{2,4\}$ and the sum of the roots is 6.

Case II: If $x^2 - 18x + 31 = 1 \rightarrow x^2 - 18x + 30 = 0$, then the sum of the roots is 18.

Case III: If $x^2 - 18x + 31 = -1 \rightarrow x^2 - 18x + 32 = 0$ and $x^2 - 6x + 8$ is even (which it is for both values of x in the solution set of the Case I equation), then the sum of the roots is 18.
Thus, the required sum is $6 + 18 + 18 = \mathbf{42}$.