Nassau County Interscholastic Mathematics League

Team Contest Answers must be integers from 0 to 999, inclusive. 2021 – 2022

Calculators are allowed.

Time: 40 minutes

31. If 3 cats can catch 3 mice in 3 days, compute the number of mice 30 cats can catch in 30 days assuming that cats catch mice at the same rate all of the time.

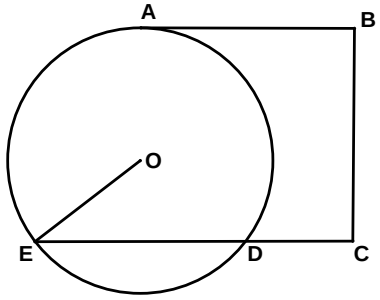
32. When $x^2 + ax + b$ is divided by $x + b$, the quotient is $x + b$ and the remainder is -20. If a and b are real numbers, compute the sum of all possible values of a.

33. Compute n if $\sum_{k=1}^{n} k = 2850$.

34. A cube is inscribed in a sphere with a radius of length 12. If the length of an edge of the cube can be expressed in simplest form as $p\sqrt{q}$, compute pq.

35. Compute the sum of all real numbers x such that the mean and median of the following five numbers are equal: $13, 16, 21, 26, x$.

36. In the given figure, $\overleftrightarrow{AB}$ is tangent to circle O at point A, $\overline{AB} \perp \overline{BC}$ and $\overline{BC} \perp \overline{EDC}$, $OE = 15$, $BC = 24$, and $CE = 36$. If the length of $\overline{AC}$ in simplest radical form is $p\sqrt{q}$, compute $p + q$.



37. The line whose equation is $y = 2x + 3$ is dilated by a scale factor of 2 about the point whose coordinates are $(1,0)$. If the equation of the image line is $y = mx + b$, compute $m + b$.

38. A point (x, y) in the coordinate plane is considered to be a lattice point if both x and y are integers. Compute the number of lattice points in the region determined by $\{(x, y): x^2 + y^2 \leq 25\}$.

39. Compute the sum of all positive integers x, such that $\frac{1}{x} = 0.\overline{pqr}$ is a repeating, non-terminating decimal whose repetend consists of exactly three distinct digits.

40. If $\log_b 10^3 + \log_b 10^4 + \log_b 10^5 + \cdots + \log_b 10^{10} = 156$ and $b = 10^{\frac{1}{n}}$, compute n.

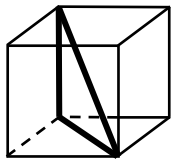
Solutions for Team Contest

31. If 3 cats can catch 3 mice in 3 days, then 30 cats can catch 30 mice in 3 days and 30 cats can catch **300** mice in 30 days.

32. From the question, $x^2 + ax + b = (x + b)^2 - 20 \rightarrow$
$x^2 + ax + b = x^2 + 2bx + b^2 - 20$. Set coefficients of like terms equal:
$a = 2b$ and $b = b^2 - 20 \rightarrow b^2 - b - 20 = 0 \rightarrow (b - 5)(b + 4) = 0 \rightarrow$
$b = 5$ or -4 and $a = 10$ or -8. The required sum is **2.**

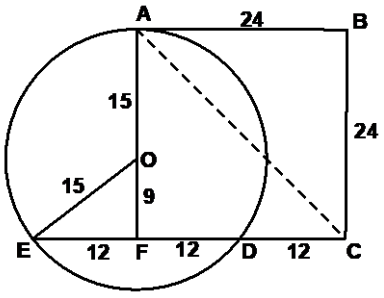
33. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = 2850 \rightarrow \frac{n(n+1)}{2} = 2850 \rightarrow n^2 + n - 5700 = 0 \rightarrow$
$(n - 75)(n + 76) = 0 \rightarrow n =$ **75**.

34. The lengths of the main diagonal of the cube and the diameter of the sphere are both 24. If the length of an edge of the cube is x, the main diagonal is $x\sqrt{3}$. So, $x\sqrt{3} = 24 \rightarrow 3x^2 = 24^2 \rightarrow x^2 = 192 \rightarrow x = \sqrt{192} = 8\sqrt{3}$. The required product is **24**.

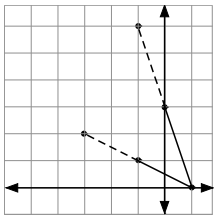


35. For the mean and median to be equal, we require the unknown number to be between the two middle numbers inclusively: $16 \leq \frac{76+x}{5} \leq 21 \rightarrow 4 \leq x \leq 29$.
If $x = 4$, the mean is 16. The numbers are 13, 16, 16, 21, 26 and the median is 16. If $x = 19$, the mean and median are each 19. If $x = 29$, the mean and median are each 21. The required sum is $4 + 19 + 29 =$ **52.** Alternatively, since the median must be 16, 21, or x, set the mean equal to each of these and solve: $\frac{76+x}{5} = 16 \rightarrow x = 4$,
$\frac{76+x}{5} = 21 \rightarrow x = 29$, $\frac{76+x}{5} = x \rightarrow x = 19$. So, $4 + 29 + 19 =$ **52**.

36. Let $\overrightarrow{AO} \cap \overline{EC} =$ point F. Since $\overrightarrow{AO} \perp \overline{AB}, \overline{AOF} \perp \overline{EFC}$, $AF = 24$, and $OF = 9$, then by the Pythagorean Theorem in ΔOEF, $EF = 12$. Then $CF = CE - EF = 36 - 12 = 24$. Note that quadrilateral $ABCF$ is a square. Therefore $AC = 24\sqrt{2}$ and the required sum is **26**.



37. Choose any two points on the given line, say, $(-1,1)$ and $(0,3)$. Their image points are $(-3,2)$ and $(-1,6)$ respectively. The slope of the image line is 2 and an equation of the image line is $y-6=2(x+1)$ or $y=2x+8$. The required sum is **10**.



38. In the given region: On the line $y=0$, $|x|\leq 5$ and there are 11 lattice points. On the line $y=1$, $|x|\leq\sqrt{24}$ and there are 9 lattice points. On the line $y=2$, $|x|\leq\sqrt{21}$ and there are 9 lattice points. On the line $y=3$, $|x|\leq 16$ and there are 9 lattice points. On the line $y=4$, $|x|\leq 9$ and there are 7 lattice points. On the line $y=5$, $|x|\leq 0$ and there is 1 lattice point. So, for $|y|\leq 5$, there are $11+2(9+9+9+7+1)=\mathbf{81}$ lattice points in the given region. Alternatively, count the number of lattice points in quadrant I and multiply that by 4. Then add in the lattice points on the axes. There are 4 lattice points above (1, 0), (2, 0) and (3, 0). There are only 3 lattice points above (4, 0). Thus, there are $4+4+4+3=15$ points in Q1. The number of lattice points on the axes is $4\cdot 5+1=21$. So, all together, there are $4(15)+21=\mathbf{81}$ lattice points satisfying the problem.

39. Multiply the given equation by 1000 to obtain $\frac{1000}{x}=pqr.\overline{pqr}$. Subtract to yield: $\frac{999}{x}=(pqr)\rightarrow x(pqr)=999$, where (pqr) is a three-digit integer with distinct digits. Since $999=3^3\cdot 37$, the only integral factors of 999 are 1, 3, 9, 27, 37, 111, 333, and 999. The only factors for which the repetend has three distinct digits are 27 (or 027) and 37 (or 037). Note that 001 has non-distinct digits. So, the only two possible values of x are $\frac{999}{27}=37$ or $\frac{999}{37}=27$. The required sum is **64**.

40. The given equation can be re-written as $3\log_b 10+4\log_b 10+5\log_b 10+\cdots+10\log_b 10=156\rightarrow 52\log_b 10=156\rightarrow\log_b 10=3\rightarrow b^3=10\rightarrow b=10^{\frac{1}{3}}\rightarrow n=\mathbf{3}$.