Nassau County Interscholastic Mathematics League

Team Contest   Answers must be integers from 0 to 999, inclusive.   2020 – 2021

Calculators are allowed.

Time: 40 minutes

31) Compute the largest three-digit number that is divisible by 3, 4, 5, and 6.

32) If $\frac{\log a}{\log b} = 512$, compute $\frac{\log\left(\frac{a}{b}\right)}{\log b}$.

33) The vertices of ΔABC are $A(-2,2)$, $B(1,-1)$, and $C(10,2)$.
Point D is equidistant from points A, B, and C. Compute the x-coordinate of point D.

34) There are two values of k for which $\binom{2020}{1020} + \binom{2020}{1021} = \binom{2021}{k}$.
Compute the positive difference of these two values of k.
Note: The notation $\binom{n}{k}$ represents the combination ${}_{\mathrm{n}}C_{\mathrm{k}}$.

35) In three dimensional Euclidean space, the coordinates of the vertices of ΔPQR are $P(5,-7,-4)$, $Q(9,-9,3)$, and $R(1,4,7)$. If the coordinates of the centroid of ΔPQR are (x,y,z), compute $30(x+y+z)$.

36) Two circles have radii of 2 and 14. One of their common internal tangents and one of their common external tangents are perpendicular. Compute the distance between their centers.

37) When the three numbers 592, 960, and 1443 are each divided by the positive integer d, the remainder in each case is r. Compute r.

38) A number palindrome is a positive integer that is the same positive integer when the order of its digits is reversed. Some examples are 515, 3663, 751157, … . If an integer is chosen at random from the set of all 19-digit positive integers and the probability that it is a number palindrome is $\frac{1}{10^k}$, compute k.

39) Compute $f\left(\frac{1}{2}\right)$ if $g(x) = 2 - x^3$ and $f(g(x)) = \frac{x^3}{2-x^{3,}}$, $x \neq \sqrt[3]{2}$.

40) In ΔRST, $RS = RT$ and $\cos(< SRT) = \frac{1}{3}$. Points P and Q trisect $\overline{ST}$.
If, in simplest form, $\cos(<PRQ) = \frac{c}{k}$, compute ck.

Hint: By the half-angle formula, if $0° < \theta < 180°$, then $\cos\frac{1}{2}\theta = \sqrt{\frac{1+\cos\theta}{2}}$.

Solutions for Team Contest

31) The LCM of 3, 4, 5, and 6 is 60. The largest three digit number that is divisible by 3, 4, 5, and 6 is 16 times 60 or **960.**

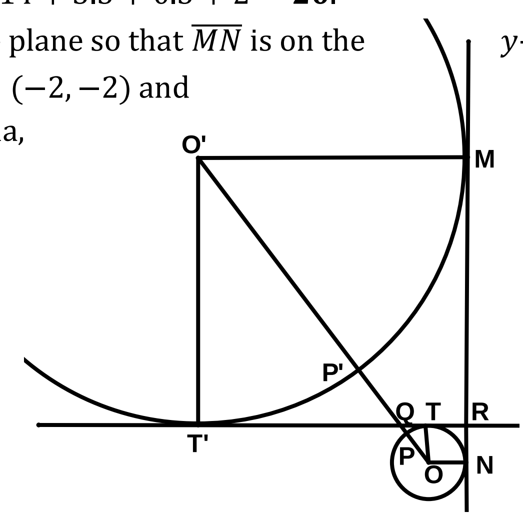
32) $\frac{\log\left(\frac{a}{b}\right)}{\log b} = \frac{\log a - \log b}{\log b} = \frac{\log a}{\log b} - \frac{\log b}{\log b} = 512 - 1 = \mathbf{511}.$

33) Point D is the intersection of the perpendicular bisectors of the sides of the triangle. The equation of the perpendicular bisector of $\overline{AC}$ is $x = \mathbf{4}.$

34) $\binom{2020}{1020} + \binom{2020}{1021} = \frac{2020!\cdot 1021}{1000!\cdot 1020!\cdot 1021} + \frac{2020!\cdot 1000}{999!\cdot 1021!\cdot 1000} = \frac{2021!}{1000!\cdot 1021!} = \binom{2021}{1021} = \binom{2021}{1000}.$
So, $1021 - 1000 = \mathbf{21}.$

35) The medians of a triangle meet at its centroid, two-thirds of the way from any vertex to its opposite midpoint. Without loss of generality, use point P as the vertex. The midpoint of $\overline{QR}$ is point M $\left(5, -\frac{5}{2}, 5\right)$. Let point C be the centroid. Then, $x_C = 5 + \frac{2}{3}(5-5) = 5$, $y_C = -7 + \frac{2}{3}\left(-\frac{5}{2} + 7\right) = -4$, and $z_C = -4 + \frac{2}{3}(5+4) = 2$. The required product is $30(5 - 4 + 2) = \mathbf{90}.$ Alternatively, the coordinates of the centroid can be found by averaging the coordinates of the triangle's vertices.

36) In the figure, the circles have centers O and O', radii $\overline{OP}$ and $\overline{O'P'}$, and points of tangency T, M, N and T'. Let point R be the intersection of the two tangents and let $QP = x$ and $QP' = 7x$. Using the Pythagorean Theorem in right triangles $O'T'Q$ and OTQ: $T'Q = \sqrt{(14+7x)^2 - 14^2}$, $TQ = \sqrt{(2+x)^2 - 2^2}$, and $T'R = T'Q + TQ + TR$. Since $T'O'MR$ and $TONR$ are squares, $T'O' = T'R = 14$ and $TO = TR = 2$. So, $14 = 7\sqrt{x^2+4x} + \sqrt{x^2+4x} + 2 \rightarrow$
$12 = 8\sqrt{x^2+4x} \rightarrow 3 = 2\sqrt{x^2+4x} \rightarrow 4x^2 + 16x - 9 = 0 \rightarrow (2x-1)(2x+9) = 0 \rightarrow$
$x = \frac{1}{2}$ only $\rightarrow OO' = O'P' + P'Q + QP + PO = 14 + 3.5 + 0.5 + 2 = \mathbf{20}.$
Alternatively, place the diagram in the coordinate plane so that $\overline{MN}$ is on the y-axis and $\overline{RT'}$ is on the x-axis. Points O and O' are $(-2,-2)$ and $(-14, 14)$, respectively. Using the distance formula, $OO' = \mathbf{20}.$



37) If two numbers leave the same remainder when divided by the same positive integer, then their difference must also be divisible by that positive integer. Note that $1443 - 960 = 483 = 23 \cdot 21$ and $960 - 592 = 368 = 23 \cdot 16$. So, if each of the given numbers is divided by 23, the remainder is **17.**

38) Since the leftmost digit may not be zero, there are nine choices for the leftmost digit and ten choices for each of the next nine digits. The last nine digits are then already determined by the first nine digits. Therefore, the required probability is $\frac{9 \cdot 10^9}{9 \cdot 10^{18}} = \frac{1}{10^9}$ and $k = \mathbf{9}$.

39) Let $g(x) = \frac{1}{2} \rightarrow \frac{1}{2} = 2 - x^3 \rightarrow x^3 = \frac{3}{2}$. So, $f\left(\frac{1}{2}\right) = \frac{\frac{3}{2}}{2 - \frac{3}{2}} = \mathbf{3}$.

Alternatively, let $n = 2 - x^3$. So, $f(n) = \frac{2-n}{n}$ and $f\left(\frac{1}{2}\right) = \mathbf{3}$.

40) Draw $\overline{RP}, \overline{RQ}$, and $\overline{RM}$, where point M is the midpoint of $\overline{ST}$. Further, let $RS = RT = a, RP = RQ = b$, and $SP = PQ = QT = c$. Note, too that $PM = MQ = \frac{c}{2}$. Also, let $RM = d$. Since $\cos(< SRT) = \frac{1}{3}$, then by the half-angle formula, $\cos(< SRM) = \sqrt{\frac{1+\frac{1}{3}}{2}} = \frac{\sqrt{2}}{\sqrt{3}}$. Let $d = RM = \sqrt{2}$, so $a = \sqrt{3}$ and $SM = \frac{3c}{2} = \sqrt{3-2} = 1$ by the Pythagorean Theorem. So, $c = \frac{2}{3}$. Since $\cos(< SRT) = \frac{1}{3}$, $\sin(< SRT) = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$. So, area$\Delta SRT$ $= |SRT| = \frac{1}{2}a^2 \sin(< SRT) = \frac{1}{2}(3)\left(\frac{2\sqrt{2}}{3}\right) = \sqrt{2}$. It follows that $= |PRQ| = \frac{1}{3}|SRT| = \frac{\sqrt{2}}{3} = \frac{1}{2}b^2 \sin(< PRQ)$. Since $b^2 = d^2 + \left(\frac{c}{2}\right)^2 = 2 + \frac{1}{9} = \frac{19}{9}$, $\sin(< PRQ) = \frac{6\sqrt{2}}{19}$. Finally, $\cos(< PRQ) = \sqrt{1 - \frac{72}{361}} = \sqrt{\frac{289}{361}} = \frac{17}{19}$ and

$17 \cdot 19 = \mathbf{323}$.