Nassau County Interscholastic Mathematics League

Team Contest     Answers must be integers from 0 to 999, inclusive.     2019 – 2020

Calculators are allowed.

**Time: 40 minutes**

31) Compute the arithmetic mean of the smallest string of 7 consecutive positive integers that are composite.

32) Compute the number of vertices of a convex polygon that has 54 diagonals.

33) The robotics club shares its winnings. The first-place prize winner gets half the money. The second-place prize winner gets three-fourths of the money that now remains. The third-place prize winner gets three-fourths of the remainder of the money. Finally, the two fourth-place prize winners share equally the remainder of the money. The fourth-place prize winners each received $12. Compute the total value of money in dollars won by all the winners.

34) In a Fibonacci-type sequence of increasing positive integers, each number after the first two is equal to the sum of the two previous numbers. If the tenth number of the sequence is 364, compute the fifth number of the sequence.

35) A semicircle with diameter $12\sqrt{3}$ is used to form a cone by bending the semicircle so that its two corners are joined and the midpoint of the diameter is the vertex of the cone. If the volume of the cone is $k\pi$, compute $k$.

36) If $a_3 = 5$ and $a_5 = 8$, and for all positive integers, $n$, $a_n + a_{n+1} + a_{n+2} = 7$,

compute $a_{2019} - a_{2020} + a_{2021}$.

37) If $a, b, c$ form a harmonic sequence, compute $\frac{b+a}{b-a} + \frac{b+c}{b-c}$. Note: A harmonic sequence is a sequence in which the reciprocals of its terms form an arithmetic sequence.

38) If $\sin^2 x = \frac{\sqrt{2}}{2}$, then $\sec^2 x$ can be expressed in simplest radical form as $p + q\sqrt{2}$. Compute $p^3 + q^3$.

39) Compute the smallest positive integer $x$ such that $2^9$ divides $x^{2019} + 1$.

40) If $n$ is a member of the set $S = \{2,3,4,5, \dots, 40\}$, compute the sum of all the values of $n$ that are members of set $S$ for which $(n-1)!$ is NOT a multiple of $n$.

## Solutions for Team Contest

31) Consider the primes less than 100. One can see that the only pair of consecutive primes in that set with a difference of $8$ is $89$ and $97$. Therefore, the string of composites is 90, 91, 92, 93, 94, 95, and 96. Their mean is **93**.

32) In a convex polygon, a diagonal can be drawn from any vertex to any other vertex except for itself and except for its two adjacent neighboring vertices. Since we do not want to count each one twice, an $n$-gon has $\frac{n}{2}(n-3)$ diagonals.
Then, $\frac{n}{2}(n-3) = 54 \to n^2 - 3n - 108 = 0 \to (n-12)(n+9) = 0 \to n = \mathbf{12}$.

33) Let $x$ represent the total winnings in dollars. Then, $\frac{1}{2}x$ goes to first place, $\frac{3}{4}\left(\frac{1}{2}x\right) = \frac{3}{8}x$ goes to second place, $\frac{3}{4}\left(\frac{1}{8}x\right) = \frac{3}{32}x$ goes to third place, and $\$24$ goes to fourth place.
Thus, $\frac{1}{2}x + \frac{3}{8}x + \frac{3}{32}x + 24 = x \to \frac{31}{32}x + 24 = x \to \frac{1}{32}x = 24 \to x = \mathbf{768}$.
Alternatively, working backwards, $2(12) = 24$. $\frac{a}{4} = 24 \to a = 96$. $\frac{b}{4} = 96 \to b = 384$.
$\frac{c}{2} = 384 \to c = 768$.

34) The numbers in the sequence can be represented as $a$, $b$, $a+b$, $a+2b$, $2a+3b$, $3a+5b$, $5a+8b$, $8a+13b$, $13a+21b$, $21a+34b$. Thus, $21a+34b = 364$. Since 364 and $21a$ are each divisible by 7, then $b$ is also divisible by 7. Thus, $b = 7$ because if $b$ is a larger multiple of 7, the tenth term would exceed 364. Therefore, $a = 6$ and the fifth term is $2(6) + 3(7) = \mathbf{33}$.

35) The semicircle becomes the circular base of the cone. The circumference of the semicircle is $6\pi\sqrt{3}$, which becomes the circumference of the circular base of the cone. So, $6\pi\sqrt{3} = 2\pi r \to r = 3\sqrt{3}$. The radius of the semicircle becomes the slant height of the cone. Then, by the Pythagorean Theorem, $h^2 + \left(3\sqrt{3}\right)^2 = \left(6\sqrt{3}\right)^2 \to h = 9$.
Therefore, the volume of the cone is $\frac{1}{3}\pi r^2 h = 81\pi \to k = \mathbf{81}$.

36) Use the constant sum that is given: $n = 3$: $5 + a_4 + 8 = 7 \to a_4 = -6$.
$n = 4$: $-6 + 8 + a_6 = 7 \to a_6 = 5$. $n = 5$: $8 + 5 + a_7 = 7 \to a_7 = -6$.
$n = 6$: $5 - 6 + a_8 = 7 \to a_8 = 8$. $n = 7$: $-6 + 8 + a_9 = 7 \to a_9 = 5$.
$n = 8$: $8 + 5 + a_{10} = 7 \to a_{10} = -6$. These terms repeat in cycles of three with $a_{3k} = 5$, $a_{3k+1} = -6$, and $a_{3k-1} = 8$. Notice that in the requested sum, the first term number is a multiple of three, the next term number is one more than a multiple of three, and the last is one less than a multiple of three.
The required sum is $5 - (-6) + 8 = \mathbf{19}$.

37) Let $a = 1$, $b = \frac{1}{2}$, and $c = \frac{1}{3}$ (Note: The reciprocals of $a, b,$ and $c$ form the arithmetic sequence, 1, 2, 3). The expression $\frac{b+a}{b-a} + \frac{b+c}{b-c} = \frac{\frac{1}{2}+1}{\frac{1}{2}-1} + \frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{2}-\frac{1}{3}} = \frac{\frac{3}{2}}{-\frac{1}{2}} + \frac{\frac{5}{6}}{\frac{1}{6}} = -3 + 5 = \mathbf{2}.$

Alternatively, from the given definition, $\frac{\frac{1}{a}+\frac{1}{c}}{2} = \frac{1}{b} \to \frac{a+c}{2ac} = \frac{1}{b} \to b = \frac{2ac}{a+c}$. The latter is known as the harmonic mean. Then, $\frac{b+a}{b-a} + \frac{b+c}{b-c} = \frac{\frac{2ac}{a+c}+a}{\frac{2ac}{a+c}-a} + \frac{\frac{2ac}{a+c}+c}{\frac{2ac}{a+c}-c} \to$
$\frac{2ac+a^2+ac}{2ac-a^2-ac} + \frac{2ac+ac+c^2}{2ac-ac-c^2} = \frac{3ac+a^2}{ac-a^2} + \frac{3ac+c^2}{ac-c^2} = \frac{3c+a}{c-a} + \frac{3a+c}{a-c} = \frac{2c-2a}{c-a} = 2.$

38) $\sec^2 x = \frac{1}{\cos^2 x} = \frac{1}{1-\sin^2 x} = \frac{1}{1-\frac{\sqrt{2}}{2}} = \frac{2}{2-\sqrt{2}} = \frac{2(2+\sqrt{2})}{2} = 2 + \sqrt{2} \to p^3 + q^3 = \mathbf{9}$.

39) Since $x^{2019} + 1$ is even, $x$ is odd. Also, $x^{2019} + 1 =$
$(x+1)(x^{2018} - x^{2017} + x^{2016} - + \cdots - x + 1)$. Note the second factor must be odd because it is the sum of an odd number (2019) of odd terms. Thus $2^9$ divides $x^{2019} + 1$ if and only if $2^9$ divides $x + 1$. Therefore, the smallest such $x$ is $2^9 - 1 = \mathbf{511}$.

40) We start experimentally: If $n$=2, $(2-1)! = 1! = 1$ is NOT a multiple of 2. If $n$=3,
$(3-1)! = 2! = 2$ is NOT a multiple of 3. If $n$=4, $(4-1)! = 3! = 6$
is NOT a multiple of 4. If $n$=5, $(5-1)! = 4! = 24$ is NOT a multiple of 5. If $n$=6,
$(6-1)! = 5! = 120$ IS a multiple of 6. If $n$=7,
$(7-1)! = 6! = 720$ is NOT a multiple of 7. If $n$=8,
$(8-1)! = 7! = 5{,}040$ IS a multiple of 8. If $n$=9,
$(9-1)! = 8! = 40{,}320$ IS a multiple of 9.
We could conjecture that if $n$ is 4 or prime that $(n-1)!$ is NOT a multiple of $n$.
We will show that there are no other possible values of $n$ that meet the criteria of the problem. Suppose that $n \neq 4$ and $n$ is not prime. Then there must be a prime $p$ such that $n = pm$. Since $n$ is not a prime, $p < n$. Thus both $p$ and $m$ are among the factors of $(n-1)!$. Specifically, if $p$ and $m$ are not the same number, then $n = pm$ is a factor of $(n-1)!$. However, if $p$ and $m$ are equal, then $n = p^2$. Thus $p > 2$ since $n \neq 4$. Thus, $2p < n$ and $p$ and $2p$ are each factors of $(n-1)!$. Thus, $(n-1)!$ is a multiple of $p \cdot 2p = 2n$ and thus is a multiple of $n$. The values of $n$ that are members of set $S$ that fit the criteria of the problem are 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 29, 31, and 37. Their sum is **201**.