Nassau County Interscholastic Mathematics League

Contest #1 Answers must be integers from 0 to 999, inclusive. 2018 – 2019

No calculators are allowed.

Time: 10 minutes

1) Compute the product of the positive square root of 1764 and the largest prime factor of 1764.

2) Compute the product of the roots of $\left|2-|x|\right|=1$.

Time: 10 minutes

3) Compute the number of cubic yards of topsoil needed to elevate a 27 foot by 27 foot garden by 4 inches.

4) Compute the smallest positive integer x such that $\sqrt{x^2+12x+35}>20$.

Time: 10 minutes

5) If $3^{50}+3^{50}+3^{50}=3^x$, compute x.

6) Diameter $\overline{AC}$ of circle O is drawn. Chord $\overline{AB}$ is extended through point B to external point D so that $AB : BD = 3 : 1$ and $\overline{DC}$ is tangent to the circle . If the ratio of the area of ΔACD to the area of ΔAOB is $\frac{p}{q}$, where $\frac{p}{q}$ is in simplest form, compute $p+q$.

<u>Solutions for Contest #1</u>

1) Since $1764 = 42^2 = (2 \cdot 3 \cdot 7)^2$, the required product is $42 \cdot 7 = \mathbf{294}$.

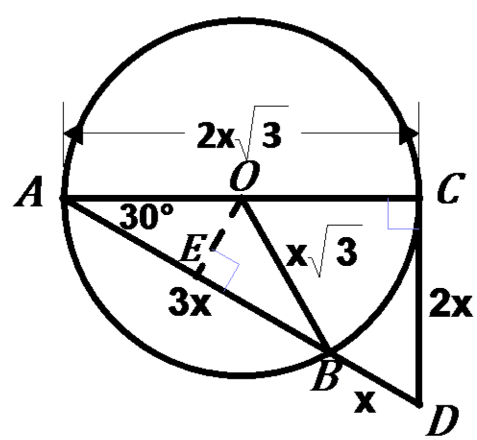
2) Either $2 - |x| = 1$ or $2 - |x| = -1$. In the first case, $x = \pm 1$. In the second case, $x = \pm 3$. The required product is **9.**

3) $27 \text{ ft} \cdot 27 \text{ ft} \cdot \frac{1}{3} \text{ ft} = 243 \text{ cu ft} \cdot \left(\frac{1 \text{ cu yd}}{27 \text{ cu ft}}\right) = \mathbf{9}$ cu yds.

4) $\sqrt{x^2 + 12x + 35} = \sqrt{(x+6)^2 - 1} > 20 \rightarrow (x+6)^2 - 1 > 400 \rightarrow x + 6 > 20 \rightarrow$ $x > 14$. Therefore the smallest positive integer to satisfy the given inequality is **15.**

5) $3^{50} + 3^{50} + 3^{50} = 3 \cdot 3^{50} = 3^{51}$. Thus, $x = \mathbf{51}$.

6) In a circle, if tangent and secant segments are drawn from the same external point, the square of the length of the tangent segment equals the product of the lengths of the secant and its external segment. So, $(CD)^2 = (4x)(x) \longrightarrow CD = 2x$. Since diameter $\overline{AC}$ is perpendicular to tangent $\overline{CD}$, and $CD = \frac{1}{2}(AD)$, ΔADC is a 30-60-90 triangle with $m\measuredangle A = 30°$ and $m\measuredangle D = 60°$. So, $AC = 2x\sqrt{3}$ and radii $AO = OB = x\sqrt{3}$. In isosceles ΔAOB, draw altitude $\overline{OE}$ whose length is $\frac{1}{2}(AO) = \frac{x\sqrt{3}}{2}$. The area of $\Delta AOB =$ $\frac{1}{2}(3x)\left(\frac{x\sqrt{3}}{2}\right) = \frac{3x^2\sqrt{3}}{4}$ The area of $\Delta ADC = \frac{1}{2}(2x)(2x\sqrt{3}) = 2x^2\sqrt{3}$. Thus, the required ratio is $\frac{8}{3}$ and $8 + 3 = \mathbf{11}$



Nassau County Interscholastic Mathematics League

Contest #2 Answers must be integers from 0 to 999, inclusive. 2018 – 2019

Calculators are allowed.

Time: 10 minutes

7) Compute the maximum number of Fridays in any calendar year.

8) Compute the sum of the series: $\frac{1}{2}+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\cdots+\left(\frac{1}{60}+\frac{2}{60}+\frac{3}{60}+\cdots+\frac{59}{60}\right)$.

Time: 10 minutes

9) Compute the sum of two consecutive positive integers whose squares differ by 95.

10) In square $WXYZ$, point M is the midpoint of $\overline{WX}$. If $WX = 13$ and point K is the intersection of $\overline{WY}$ and $\overline{ZM}$, compute $\frac{ZK}{KM}$.

Time: 10 minutes

11) Compute the number of three-digit numbers such that the units digit is larger than the tens digit.

12) In a rectangular coordinate system, a line contains the point whose coordinates are $(8,10)$. The line's x-intercept is twice its y-intercept. The area of the circle that circumscribes the triangle formed by the line and the coordinate axes is $k\pi$. Compute k.

Solutions for Contest #2

7) A year contains either $365 = 7 \cdot 52 + 1$ or $366 = 7 \cdot 52 + 2$ days. Therefore, a year must contain at least 52 Fridays. Of the remaining 1 or 2 days, no more than 1 may be a Friday. Therefore, there are at most **53** Fridays in a year.

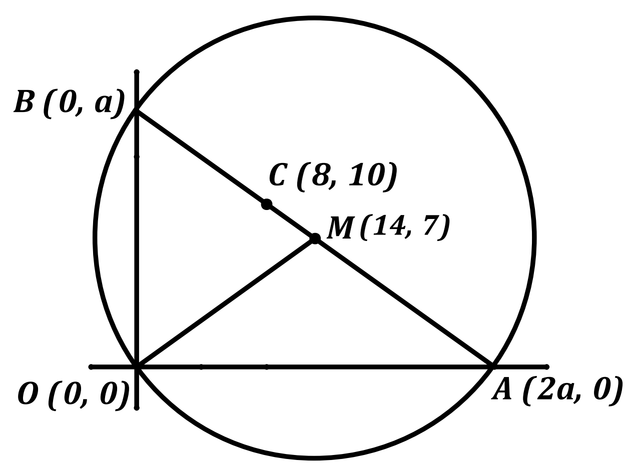
8) Use the fact that $1 + 2 + 3 + \cdots + 58 + 59 = \frac{59 \cdot 60}{2} = 1770$ to rewrite the given sum as $\frac{1}{2} + \frac{3}{3} + \frac{6}{4} + \frac{10}{5} + \cdots + \frac{1770}{60} = \frac{1}{2} + 1 + \frac{3}{2} + 2 + \cdots + \frac{59}{2} = \frac{1}{2}(1 + 2 + 3 + \cdots + 59) = \mathbf{885.}$

9) The integers are x and $x + 1$. Their sum is $2x + 1$. Then, $(x + 1)^2 - x^2 = 95 \rightarrow$ $2x + 1 = \mathbf{95}$.

10) Notice that $\Delta ZYK \sim \Delta MWK \rightarrow \frac{ZY}{MW} = \frac{ZK}{KM} = \frac{2}{1} = \mathbf{2.}$

11) There are 100 2-digit numbers (00, 01, 02, … , 99) that may be appended to a hundreds digit to form a 3-digit number. Ten of these 2-digit numbers have the same digits (00, 11, 22, … , 99), so of the remaining 90 2-digit numbers, 45 (half) of them have their units digit greater than their tens digit. Since there are 9 possible hundreds digits, $9 \cdot 45 = \mathbf{405}$.

12) As in the diagram, let the coordinates of O be $(0,0)$, A be $(2a, 0)$, B be $(0, a)$, and C be $(8, 10)$. Since the slope of $\overline{AB} = \frac{a-0}{0-2a} = -\frac{1}{2}$, and points A, B, and C are collinear, the slope of $\overline{BC} = \frac{10-a}{8-0} = -\frac{1}{2} \longrightarrow a = 14$. For a right triangle, the center of its circumscribed circle, the circumcenter, is the midpoint of its hypotenuse. The coordinates of midpoint M are $(14, 7)$ and by the distance formula, the length of radius $\overline{OM}$ is $\sqrt{14^2 + 7^2} = \sqrt{245}$. Thus, the area of the circle is 245π and $k = \mathbf{245}$.



Nassau County Interscholastic Mathematics League

Contest #3 Answers must be integers from 0 to 999, inclusive. 2018 – 2019

No calculators are allowed.

Time: 10 minutes

13) Compute $\left[\sqrt[3]{\sqrt{130}-\sqrt{5}}\right] \cdot \left[\sqrt[3]{\sqrt{130}+\sqrt{5}}\right]$.

14) Compute the middle term of an arithmetic sequence of 57 terms whose sum is 4845.

Time: 10 minutes

15) Compute the number of integers between 5000 and 8000 that are squares of integers.

16) In rectangle $ABCD$, $AB = 4$ and $AD = 2$. Point E is on $\overline{CD}$ such that $AE = AB$. When BE is expressed in the form $\sqrt{x} - \sqrt{y}$, compute $x - y$.

Time: 10 minutes

17) Three cuts are made through a large cube to create 8 identical smaller cubes. The surface area of the large cube is 96 cm^2. Compute the number of cm^2 in the total surface area of the 8 smaller cubes.

18) A fourth degree polynomial of the form $x^4 + cx^2 + d$, where c and d are integers, has $\sqrt{3} + i$ as a zero. Compute $c + d$.

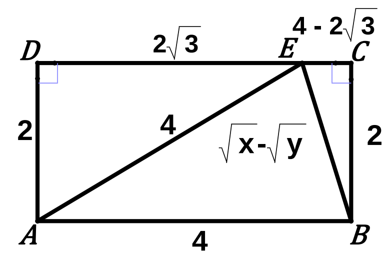
Solutions for Contest #3

13) $\left[\sqrt[3]{\sqrt{130}-\sqrt{5}}\right]\cdot\left[\sqrt[3]{\sqrt{130}+\sqrt{5}}\right]=\sqrt[3]{130-5}=\sqrt[3]{125}=\mathbf{5}.$

14) If x is the middle term and d is the common difference, then the given sequence can be expressed as $x-28d, x-27d, \dots, x-d, x, x+d, \dots, x+27d, x+28d$. Using the formula $S_n=\frac{n}{2}(a_1+a_n)$, $4845=\frac{57}{2}(x-28d+x+28d)\longrightarrow 4845=\frac{57}{2}(2x)\longrightarrow$ $4845=57x\longrightarrow x=\mathbf{85}$. Note that in an arithmetic sequence with an odd number of terms, the product of the number of terms and the middle term equals its sum.

15) Start with $70^2=4900$ and $71^2=5041$. Also, $90^2=8100$ and $89^2=7921$. The squares of the integers in the specified interval are the squares of the integers from 71 through 89. There are **19** of them.

16) Use the Pythagorean Theorem in ΔADE (or note that ΔADE is a 30-60-90 triangle) to get $DE=2\sqrt{3}$. Then $CE=4-2\sqrt{3}$. Let $BE=\sqrt{x}-\sqrt{y}$ and use the Pythagorean Theorem again in ΔBCE to get $\left(\sqrt{x}-\sqrt{y}\right)^2=4+\left(4-2\sqrt{3}\right)^2\longrightarrow x+y-2\sqrt{xy}=$ $32-16\sqrt{3}\longrightarrow x+y-2\sqrt{xy}=32-2\sqrt{192}$. This yields the system of equations: $x+y=32$ and $xy=192\longrightarrow$ $x(32-x)=192\longrightarrow x^2-32x+192=0\longrightarrow$ $(x-24)(x-8)=0$. Since $x>y$, $(x,y)=(24,8)$ and the required difference is 1**6.**



17) Since the ratio of the volumes of a small cube to the large cube is 1 : 8, the ratio of their edges is 1 : 2. Therefore, the surface area of each small cube is one-fourth the surface area of the large cube. Since there are 8 small cubes, their total surface area is $8\cdot\frac{1}{4}=2$ times the surface area of the large cube. That total surface area is **192** cm^2.

18) Since complex zeros of polynomials with integral coefficients occur in conjugate pairs, $\sqrt{3}-i$ is also a zero. Using the sum and product of these zeros, the quadratic function with these two zeros is $x^2-2\sqrt{3}x+4$. Since we are looking for a fourth degree polynomial, we try squaring this quadratic. Doing so, however, will not make all of the coefficients integral because the cubic and linear terms are not eliminated. By some trial and error, noting that all four zeros have a sum of 0, and/or deducing that $-\sqrt{3}+i$ and $-\sqrt{3}-i$ are also zeros, we discover that multiplying the quadratic by $x^2+2\sqrt{3}x+4$ gives us x^4-4x^2+16. Thus, $c+d=-4+16=\mathbf{12}$.

Nassau County Interscholastic Mathematics League

Contest #4 Answers must be integers from 0 to 999, inclusive. 2018 – 2019

Calculators are allowed.

Time: 10 minutes

19) Kevin is very frugal. On a recent purchase, he received successive discounts on the list price of 10%, 20%, and 25%. An equivalent single discount on the list price is x%. Compute x.

20) Compute the number represented by the three right-most digits of 5^{50}.

Time: 10 minutes

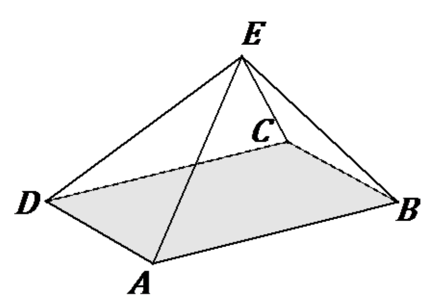
21) Compute the greatest number of mailboxes needed to hold 45 letters if each mailbox contains at least one letter and no two mailboxes contain the same number of letters.

22) The ordered triple (x, y, z) satisfies the system of equations: $x + y + z = 3$, $xy + yz + xz = -1$, $xyz = -6$. If the maximum value of $x + y$ is expressed in simplest $\frac{a+\sqrt{b}}{c}$ form, compute abc.

Time: 10 minutes

23) A sequence of terms is defined as follows: $a_1 = 1$, $a_2 = 3$, and $a_n = a_{n-1} - a_{n-2}$. Compute $a_1 + a_2 + a_3 + \cdots + a_{198} + a_{199} + a_{200}$.

24) Point E is not in the plane of square $ABCD$ so that $AE = BE = CE = DE = AB$. Point M is the midpoint of $\overline{AE}$ and point N is the midpoint of $\overline{BE}$. Point F is the foot of the perpendicular from point E to the plane containing square $ABCD$. The plane containing points C, M, and N intersects $\overline{EF}$ in point G. Compute $12 \cdot \frac{EG}{GF}$.



Solutions for Contest #4

19) Without loss of generality, assume the list price is \$100. Then the 10% discount yields a price of \$90, followed by a 20% discount yielding a price of \$72, followed by a 25% discount yielding a price of \$54. Thus, an equivalent single discount is 46% and $x = \mathbf{46}$**.**

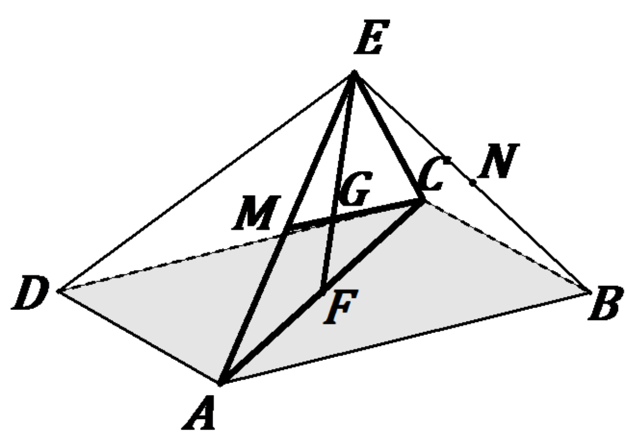
20) Consider powers of 5 mod 1000: $5^1 \equiv 5, 5^2 \equiv 25, 5^3 \equiv 125, 5^4 \equiv 625, 5^5 \equiv 125,$ $5^6 \equiv 625, \ldots$. For all even exponent powers of 5, where the exponent is greater than 3, the three rightmost digits of the numeral is **625**.

21) Start summing consecutive integers and note that 1+2+3+4+5+6+7+8+9 = 45. Hence, **9** mailboxes are needed.

22) The given system of equations reveals that the sum of the roots of a cubic equation is 3, the sum of the product of the roots taken two at a time is -1, and the product of the roots is -6. Therefore, $x, y,$ and z are the roots of $w^3 - 3w^2 - w + 6 = 0 \rightarrow$ $(w-2)(w^2 - w - 3) = 0 \rightarrow w = 2, \frac{1+\sqrt{13}}{2}, \frac{1-\sqrt{13}}{2}$. Each of the permutations of these three roots corresponds to an ordered triple that satisfies the system. The requested maximum is $2 + \frac{1+\sqrt{13}}{2} = \frac{5+\sqrt{13}}{2}$ and $abc = 5 \cdot 13 \cdot 2 = \mathbf{130}$**.**

23) Note that $a_1 = 1, a_2 = 3, a_3 = 2, a_4 = -1, a_5 = -3, a_6 = -2, a_7 = 1, a_8 = 3, a_9 = 2, \ldots$. There is a repetition of the sequence 1, 3, 2, -1, -3, -2 every six terms and the sum of the repeating sequence is 0. Since 6 is a divisor of 198, $\sum_{k=1}^{198} a_k = 0$ and $\sum_{k=1}^{200} a_k = 0 + a_{199} + a_{200} = 0 + 1 + 3 = \mathbf{4}$**.**

24) Point F is at the center of square $ABCD$ and is the midpoint of diagonal $\overline{AC}$. Therefore, $\overline{EF}$ and $\overline{CM}$ are medians in ΔEAC that meet at point G which divides the medians into segments whose lengths are in a 2:1 ratio. So, $\frac{EG}{GF} = \frac{2}{1}$ and the required answer is $12 \cdot 2 = \mathbf{24}$.



Nassau County Interscholastic Mathematics League

Contest #5 Answers must be integers from 0 to 999, inclusive. 2018 – 2019

No calculators are allowed.

Time: 10 minutes

25) Compute the units digit of the product of all prime numbers less than or equal to 120.

26) Five men, working together, who each plow at the same uniform rate, can plow a square field whose side is 80 feet in 4 hours. At the same rate of work, compute the number of hours that 10 men could plow a square field whose side is 160 feet.

Time: 10 minutes

27) Two numbers have a sum of 9 and a product of 13. If the sum of the reciprocals of the numbers is expressed in simplest form as $\frac{p}{q}$, compute $p + q$.

28) The legs of a right triangle have lengths 9 and 40. The area of the circle inscribed in this triangle may be expressed in the form $k\pi$. Compute k.

Time: 10 minutes

29) Compute the number of pairs of positive integers (x, y) that satisfy $4x + 5y = 600$.

30) Compute the product of the roots of $x^{0.5(\log_7 x - 1)} = 7$.

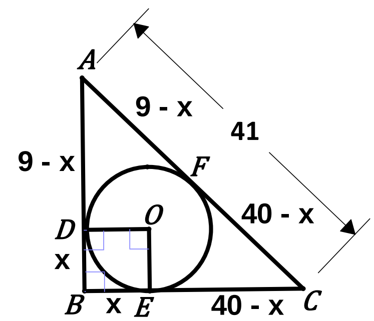
Solutions for Contest #5

25) Since the set of prime numbers less than or equal to 120 contain the numbers 2 and 5, 10 divides the product. So, its units digit is **0**.

26) To plow the field whose area is 6400 square feet requires 20 man-hours of work. Their uniform rate of work is $6400/20 = 320$ square feet per man-hour. If 10 men work at the same rate for h hours, then we require $\frac{160^2}{10h} = 320 \rightarrow \frac{16}{h} = 2 \rightarrow h = \mathbf{8}$.

27) Denote the numbers as x and y. Then, $\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{9}{13}$. The required sum is $9 + 13 = \mathbf{22}$.

28) Since $\measuredangle B$ is a right angle, $\overline{OD} \perp \overline{AB}$, $\overline{OE} \perp \overline{BC}$, and radii $OD = OE$, $BEOD$ is a square. If $OD = BD = BE = OE = x$, then $AD = 9 - x$, and $EC = 40 - x$. Since tangent segments drawn to a circle from an external point are congruent, $AF = 9 - x$ and $CF = 40 - x$. Using the Pythagorean triple 9-40-41 in ΔABC, $AC = 41$. So, $9 - x + 40 - x = 41 \longrightarrow 49 - 2x = 41 \longrightarrow x = 4$. The area of circle O is 16π. Thus, $k = \mathbf{16}$. [Note: The pattern for right triangles of given side lengths and the lengths of the radii, r, of their inscribed circles is as follows: 3,4,5: r=1; 5,12,13: r=2; 7,24,25: r=3; 9,40,41: r=4.]



29) From the given equation: $y = \frac{600-4x}{5} = 120 - \frac{4}{5}x$ and y is a positive integer. So, possible values of x are multiples of 5 from 5 to 145, inclusive. Thus, there are $145/5 = \mathbf{29}$ ordered pairs.

30) Take logs of both sides of the given equation using base 7:
$\log_7 x^{0.5(\log_7 x - 1)} = 1 \rightarrow \frac{1}{2}(\log_7 x - 1)\log_7 x = 1$. Let $y = \log_7 x$.
Then, $\frac{1}{2}(y-1)y = 1 \rightarrow y^2 - y - 2 = 0 \rightarrow y = 2$ or $y = -1 \rightarrow x = 49$ or $\frac{1}{7}$.
The required product is $49 \cdot \frac{1}{7} = \mathbf{7}$.

Nassau County Interscholastic Mathematics League

Team Contest Answers must be integers from 0 to 999, inclusive. 2018 – 2019

Calculators are allowed.

Time: 40 minutes

31) Compute the smallest positive perfect square integer that is divisible by 3, 4, and 5.

32) When $2+\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}}$ is expressed in simplest $\frac{p}{q}$ form, compute $p+q$.

33) Compute the sum of the roots of $7\sqrt{x}-x-12=0$.

34) Tom can dig a rectangular ditch 6 feet long, 2 feet wide and 5 feet deep in 30 minutes. If his twin brother Jerry, working at the same rate, works together with Tom, compute the number of minutes it would take them to dig a ditch 12 feet long, 4 feet wide, and 10 feet deep.

35) Compute the number of lattice points in the set $\{(x,y): x^2+y^2 \leq 25\}$. By definition, a lattice point is a point in the Cartesian plane such that its coordinates are both integers.

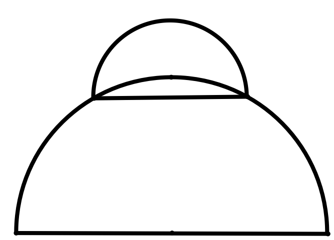
36) Compute the first term of an infinite geometric sequence if the sum of the series is 4 and the sum of the cubes of the terms of the series is 192.

37) Compute the volume of a pyramid whose base is an equilateral triangle with side lengths that are each 24, and whose other edges are each $4\sqrt{15}$.

38) The sum of three terms of an arithmetic sequence is 12 and the sum of their cubes is 408. Compute the sum of the squares of the three terms.

39) A set consists of only non-negative integers. No matter what pair of distinct numbers is selected from the set, neither their sum nor their difference will be a multiple of 10. Compute the maximum number of elements in this set.

40) A semicircle with a diameter of length 12 sits atop a semicircle with a diameter of length 24 as shown. The area interior to the smaller semicircle and exterior to the larger semicircle is called a lune. If the area of this lune is expressed as $a\sqrt{3}-b\pi$, compute ab.



Solutions for Team Contest

31) The number must be divisible by 4 or 2^2, as well as 3^2 and 5^2. The product $2^2 \cdot 3^2 \cdot 5^2 = \mathbf{900.}$

32) $2 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}} = 2 + \frac{1}{2+\frac{1}{2+\frac{2}{5}}} = 2 + \frac{1}{2+\frac{5}{12}} = 2 + \frac{12}{29} = \frac{70}{29}$. The required sum is **99.**

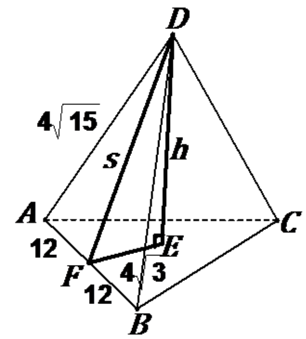
33) If $7\sqrt{x} = x + 12$, then by squaring both sides, $49x = x^2 + 24x + 144 \longrightarrow$ $x^2 - 25x + 144 = 0$. Since $-\frac{b}{a}$ is the sum of the roots of $ax^2 + bx + c = 0$, the sum of the roots of the given equation is **25**. Alternatively, let $a = \sqrt{x}$, $a \geq 0$. Then, $7a - a^2 - 12 = 0 \rightarrow a^2 - 7a + 12 = 0 \rightarrow (a-4)(a-3) = 0 \rightarrow$ $a = 4$ or $a = 3$. The roots of the given equation are 9 and 16. Their sum is **25**.

34) Each of the dimensions of the larger ditch is double the corresponding dimension of the smaller. So, the volume of the larger is 8 times the volume of the smaller. Therefore, each of the brothers will take $30 \cdot 8 = 240$ minutes working alone. Working together, they will take 240/2=**120** minutes.

35) If $y = \pm 5$, there are 2 lattice points in the given set. If $y = \pm 4$, there are 14. If $y = \pm 3$, there are 18. If $y = \pm 2$, there are 18. If $y = \pm 1$, there are 18. If $y = 0$, there are 11. Thus, $2 + 14 + 18 + 18 + 18 + 11 = \mathbf{81}$. Alternatively, count the number of lattice points in Quadrant I (15), multiply by 4, and add the number of lattice points on the coordinate axes (21). Thus, $4(15) + 21 = \mathbf{81}$.

36) The terms of the first sequence are denoted by: $a, ar, ar^2, \ldots$ and the terms of the second sequence are denoted by $a^3, a^3r^3, a^3r^6, \ldots$. Therefore, $a + ar + ar^2 + \cdots =$ $a(1 + r + r^2 + \cdots) = \frac{a}{1-r} = 4$. Also, $a^3 + a^3r^3 + a^3r^6 + \cdots = a^3(1 + r^3 + r^6 + \cdots) =$ $\frac{a^3}{1-r^3} = \frac{a^3}{(1-r)(1+r+r^2)} = \frac{a}{(1-r)} \cdot \frac{a^2}{(1+r+r^2)} = 4\left(\frac{a^2}{(1+r+r^2)}\right) = 192 \rightarrow \frac{a^2}{(1+r+r^2)} = 48$. But, $a^2 = 16(1 - 2r + r^2)$. So, $\frac{16(1-2r+r^2)}{(1+r+r^2)} = 48 \rightarrow 1 - 2r + r^2 = 3(1 + r + r^2) \rightarrow$ $2r^2 + 5r + 2 = 0 \rightarrow r = -\frac{1}{2}$ or $r = -2$. The latter r is rejected. Then, using $\frac{a}{1-r} = 4$, $a = \mathbf{6}$.

37) The volume of a pyramid is one-third the area of its base times its altitude. In the diagram, s is a slant height and h is the altitude of the pyramid from point D. The area of the pyramid's base, equilateral ΔABC is $\frac{24^2\sqrt{3}}{4} = 144\sqrt{3}$. By the Pythagorean Theorem in ΔADF, $s^2 + 12^2 = (4\sqrt{15})^2 \longrightarrow s^2 = 96 \longrightarrow s = 4\sqrt{6}$. Since point E is the center of the equilateral triangular base, $EF = \frac{1}{3}CF = \frac{1}{3}(12\sqrt{3})$ $= 4\sqrt{3}$. By the Pythagorean Theorem again, $h^2 + (4\sqrt{3})^2 = (4\sqrt{6})^2$ $\longrightarrow h^2 = 48 \longrightarrow h = 4\sqrt{3}$. So, the volume of the pyramid is $\frac{1}{3}(144\sqrt{3})(4\sqrt{3}) = \mathbf{576}$.



38) Denote the terms as $4-d, 4$, and $4+d$. Then, $(4-d)^3 + 64 + (4+d)^3 = 408 \rightarrow$ $64 - 48d + 12d^2 - d^3 + 64 + 64 + 48d + 12d^2 + d^3 = 408 \rightarrow 192 + 24d^2 = 408 \rightarrow$ $d^2 = 9 \rightarrow d = \pm 3$. The terms can be either 1, 4, and 7 or 7, 4, and 1. Either way, the sum of their squares is $1 + 16 + 49 = \mathbf{66}$. Alternatively, the problem can be solved by inspection using small perfect cubes: 1, 8, 27, 64, 125, 216, and 343. That is, by noting that $1 + 64 + 343 = 408$, their cube roots, 1, 4, and 7 lead to the correct answer.

39) Start constructing the set with non-negative integers whose units digits are 0 and 1 and build the set one number at a time until it contains non-negative integers whose units digits are 0, 1, 2, 3, 4, and 5. Note that by choosing any pair of these numbers, we satisfy the problem's requirements. No matter what additional non-negative integer we insert, we have a violation. So, the answer is **6.**

40) Since the radius of the larger semicircle is equal to the diameter of the smaller semicircle, the segment's area (the area bounded by $\overline{AB}$ and arc ACB) is equal to the area of the 60 degree sector AOB minus the area of the equilateral triangle AOB whose side length is 12. So, the area of the lune is equal to the area of the smaller semicircle minus the area of the segment: $\frac{1}{2}\pi\left(\frac{12}{2}\right)^2 - \left(\frac{1}{6}\pi \cdot 12^2 - \frac{12^2}{4}\sqrt{3}\right) =$ $18\pi - 24\pi + 36\sqrt{3} = 36\sqrt{3} - 6\pi$. The required product is **216.**