Nassau County Interscholastic Mathematics League

Team Contest   Answers must be integers from 0 to 999, inclusive.   2018 – 2019

Calculators are allowed.

Time: 40 minutes

31) Compute the smallest positive perfect square integer that is divisible by 3, 4, and 5.

32) When $2+\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}}$ is expressed in simplest $\frac{p}{q}$ form, compute $p+q$.

33) Compute the sum of the roots of $7\sqrt{x}-x-12=0$.

34) Tom can dig a rectangular ditch 6 feet long, 2 feet wide and 5 feet deep in 30 minutes. If his twin brother Jerry, working at the same rate, works together with Tom, compute the number of minutes it would take them to dig a ditch 12 feet long, 4 feet wide, and 10 feet deep.

35) Compute the number of lattice points in the set $\{(x,y): x^2+y^2 \leq 25\}$. By definition, a lattice point is a point in the Cartesian plane such that its coordinates are both integers.

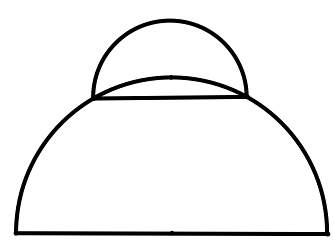
36) Compute the first term of an infinite geometric sequence if the sum of the series is 4 and the sum of the cubes of the terms of the series is 192.

37) Compute the volume of a pyramid whose base is an equilateral triangle with side lengths that are each 24, and whose other edges are each $4\sqrt{15}$.

38) The sum of three terms of an arithmetic sequence is 12 and the sum of their cubes is 408. Compute the sum of the squares of the three terms.

39) A set consists of only non-negative integers. No matter what pair of distinct numbers is selected from the set, neither their sum nor their difference will be a multiple of 10. Compute the maximum number of elements in this set.

40) A semicircle with a diameter of length 12 sits atop a semicircle with a diameter of length 24 as shown. The area interior to the smaller semicircle and exterior to the larger semicircle is called a lune. If the area of this lune is expressed as $a\sqrt{3}-b\pi$, compute ab.



Solutions for Team Contest

31) The number must be divisible by 4 or 2^2, as well as 3^2 and 5^2. The product $2^2 \cdot 3^2 \cdot 5^2 =$ **900.**

32) $2+\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}} = 2+\frac{1}{2+\frac{1}{2+\frac{2}{5}}} = 2+\frac{1}{2+\frac{5}{12}} = 2+\frac{12}{29} = \frac{70}{29}$. The required sum is **99**.

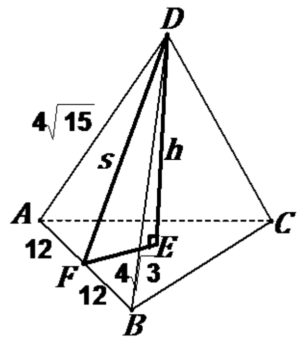
33) If $7\sqrt{x} = x + 12$, then by squaring both sides, $49x = x^2 + 24x + 144 \longrightarrow$ $x^2 - 25x + 144 = 0$. Since $-\frac{b}{a}$ is the sum of the roots of $ax^2 + bx + c = 0$, the sum of the roots of the given equation is **25**. Alternatively, let $a = \sqrt{x}$, $a \geq 0$. Then, $7a - a^2 - 12 = 0 \rightarrow a^2 - 7a + 12 = 0 \rightarrow (a-4)(a-3) = 0 \rightarrow$ $a = 4$ or $a = 3$. The roots of the given equation are 9 and 16. Their sum is **25**.

34) Each of the dimensions of the larger ditch is double the corresponding dimension of the smaller. So, the volume of the larger is 8 times the volume of the smaller. Therefore, each of the brothers will take $30 \cdot 8 = 240$ minutes working alone. Working together, they will take 240/2=**120** minutes.

35) If $y = \pm 5$, there are 2 lattice points in the given set. If $y = \pm 4$, there are 14. If $y = \pm 3$, there are 18. If $y = \pm 2$, there are 18. If $y = \pm 1$, there are 18. If $y = 0$, there are 11. Thus, $2 + 14 + 18 + 18 + 18 + 11 =$ **81**. Alternatively, count the number of lattice points in Quadrant I (15), multiply by 4, and add the number of lattice points on the coordinate axes (21). Thus, $4(15) + 21 =$ **81**.

36) The terms of the first sequence are denoted by: $a, ar, ar^2, \ldots$ and the terms of the second sequence are denoted by $a^3, a^3r^3, a^3r^6, \ldots$. Therefore, $a + ar + ar^2 + \cdots =$ $a(1 + r + r^2 + \cdots) = \frac{a}{1-r} = 4$. Also, $a^3 + a^3r^3 + a^3r^6 + \cdots = a^3(1 + r^3 + r^6 + \cdots) =$ $\frac{a^3}{1-r^3} = \frac{a^3}{(1-r)(1+r+r^2)} = \frac{a}{(1-r)} \cdot \frac{a^2}{(1+r+r^2)} = 4\left(\frac{a^2}{(1+r+r^2)}\right) = 192 \rightarrow \frac{a^2}{(1+r+r^2)} = 48$. But, $a^2 = 16(1 - 2r + r^2)$. So, $\frac{16(1-2r+r^2)}{(1+r+r^2)} = 48 \rightarrow 1 - 2r + r^2 = 3(1 + r + r^2) \rightarrow$ $2r^2 + 5r + 2 = 0 \rightarrow r = -\frac{1}{2}$ or $r = -2$. The latter r is rejected. Then, using $\frac{a}{1-r} = 4$, $a =$ **6**.

37) The volume of a pyramid is one-third the area of its base times its altitude. In the diagram, s is a slant height and h is the altitude of the pyramid from point D. The area of the pyramid's base, equilateral ΔABC is $\frac{24^2\sqrt{3}}{4} = 144\sqrt{3}$. By the Pythagorean Theorem in ΔADF, $s^2 + 12^2 = \left(4\sqrt{15}\right)^2 \longrightarrow s^2 = 96 \longrightarrow s = 4\sqrt{6}$. Since point E is the center of the equilateral triangular base, $EF = \frac{1}{3}CF = \frac{1}{3}\left(12\sqrt{3}\right) = 4\sqrt{3}$. By the Pythagorean Theorem again, $h^2 + \left(4\sqrt{3}\right)^2 = \left(4\sqrt{6}\right)^2 \longrightarrow h^2 = 48 \longrightarrow h = 4\sqrt{3}$. So, the volume of the pyramid is $\frac{1}{3}\left(144\sqrt{3}\right)\left(4\sqrt{3}\right) = \mathbf{576}$.



38) Denote the terms as $4 - d, 4$, and $4 + d$. Then, $(4 - d)^3 + 64 + (4 + d)^3 = 408 \rightarrow 64 - 48d + 12d^2 - d^3 + 64 + 64 + 48d + 12d^2 + d^3 = 408 \rightarrow 192 + 24d^2 = 408 \rightarrow d^2 = 9 \rightarrow d = \pm 3$. The terms can be either 1, 4, and 7 or 7, 4, and 1. Either way, the sum of their squares is $1 + 16 + 49 = \mathbf{66}$. Alternatively, the problem can be solved by inspection using small perfect cubes: 1, 8, 27, 64, 125, 216, and 343. That is, by noting that $1 + 64 + 343 = 408$, their cube roots, 1, 4, and 7 lead to the correct answer.

39) Start constructing the set with non-negative integers whose units digits are 0 and 1 and build the set one number at a time until it contains non-negative integers whose units digits are 0, 1, 2, 3, 4, and 5. Note that by choosing any pair of these numbers, we satisfy the problem's requirements. No matter what additional non-negative integer we insert, we have a violation. So, the answer is **6**.

40) Since the radius of the larger semicircle is equal to the diameter of the smaller semicircle, the segment's area (the area bounded by $\overline{AB}$ and arc ACB) is equal to the area of the 60 degree sector AOB minus the area of the equilateral triangle AOB whose side length is 12. So, the area of the lune is equal to the area of the smaller semicircle minus the area of the segment: $\frac{1}{2}\pi\left(\frac{12}{2}\right)^2 - \left(\frac{1}{6}\pi \cdot 12^2 - \frac{12^2}{4}\sqrt{3}\right) = 18\pi - 24\pi + 36\sqrt{3} = 36\sqrt{3} - 6\pi$. The required product is **216**.