Nassau County Interscholastic Mathematics League

Contest #4 Answers must be integers from 0 to 999, inclusive. 2018 – 2019

Calculators are allowed.

Time: 10 minutes **Name: _____**

19) Kevin is very frugal. On a recent purchase, he received successive discounts on the list price of 10%, 20%, and 25%. An equivalent single discount on the list price is x%. Compute x.

20) Compute the number represented by the three right-most digits of 5^{50}.

19. ☐ 20. ☐

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Time: 10 minutes **Name: _____**

21) Compute the greatest number of mailboxes needed to hold 45 letters if each mailbox contains at least one letter and no two mailboxes contain the same number of letters.

22) The ordered triple (x, y, z) satisfies the system of equations: $x + y + z = 3$, $xy + yz + xz = -1$, $xyz = -6$. If the maximum value of $x + y$ is expressed in simplest $\frac{a+\sqrt{b}}{c}$ form, compute abc.

21. ☐

22. ☐

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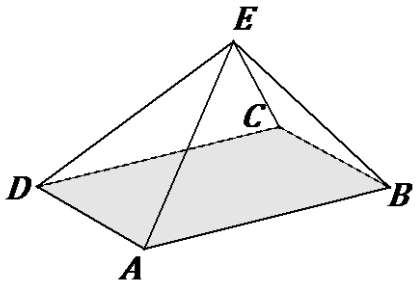
23) A sequence of terms is defined as follows: $a_1 = 1$, $a_2 = 3$, and $a_n = a_{n-1} - a_{n-2}$. Compute $a_1 + a_2 + a_3 + \cdots + a_{198} + a_{199} + a_{200}$.

24) Point E is not in the plane of square $ABCD$ so that $AE = BE = CE = DE = AB$. Point M is the midpoint of $\overline{AE}$ and point N is the midpoint of $\overline{BE}$. Point F is the foot of the perpendicular from point E to the plane containing square $ABCD$. The plane containing points C, M, and N intersects $\overline{EF}$ in point G. Compute $12 \cdot \frac{EG}{GF}$.



23. ☐

24. ☐

Solutions for Contest #4

19) Without loss of generality, assume the list price is \$100. Then the 10% discount yields a price of \$90, followed by a 20% discount yielding a price of \$72, followed by a 25% discount yielding a price of \$54. Thus, an equivalent single discount is 46% and $x = \mathbf{46.}$

20) Consider powers of 5 mod 1000: $5^1 \equiv 5, 5^2 \equiv 25, 5^3 \equiv 125, 5^4 \equiv 625, 5^5 \equiv 125, 5^6 \equiv 625, \ldots$. For all even exponent powers of 5, where the exponent is greater than 3, the three rightmost digits of the numeral is **625**.

21) Start summing consecutive integers and note that $1+2+3+4+5+6+7+8+9 = 45$. Hence, **9** mailboxes are needed.

22) The given system of equations reveals that the sum of the roots of a cubic equation is 3, the sum of the product of the roots taken two at a time is -1, and the product of the roots is -6. Therefore, $x, y,$ and z are the roots of $w^3 - 3w^2 - w + 6 = 0 \rightarrow (w-2)(w^2 - w - 3) = 0 \rightarrow w = 2, \frac{1+\sqrt{13}}{2}, \frac{1-\sqrt{13}}{2}$. Each of the permutations of these three roots corresponds to an ordered triple that satisfies the system. The requested maximum is $2 + \frac{1+\sqrt{13}}{2} = \frac{5+\sqrt{13}}{2}$ and $abc = 5 \cdot 13 \cdot 2 = \mathbf{130.}$

23) Note that $a_1 = 1, a_2 = 3, a_3 = 2, a_4 = -1, a_5 = -3, a_6 = -2, a_7 = 1, a_8 = 3, a_9 = 2, \ldots$. There is a repetition of the sequence 1, 3, 2, -1, -3, -2 every six terms and the sum of the repeating sequence is 0. Since 6 is a divisor of 198, $\sum_{k=1}^{198} a_k = 0$ and $\sum_{k=1}^{200} a_k = 0 + a_{199} + a_{200} = 0 + 1 + 3 = \mathbf{4.}$

24) Point F is at the center of square $ABCD$ and is the midpoint of diagonal $\overline{AC}$. Therefore, $\overline{EF}$ and $\overline{CM}$ are medians in ΔEAC that meet at point G which divides the medians into segments whose lengths are in a 2:1 ratio. So, $\frac{EG}{GF} = \frac{2}{1}$ and the required answer is $12 \cdot 2 = \mathbf{24}$.