Nassau County Interscholastic Mathematics League

Contest #1 Answers must be integers from 0 to 999, inclusive. 2017 – 2018

No calculators are allowed.

Time: 10 minutes

1) If $N = 20 \cdot 30 \cdot 50 \cdot 70 \cdot 110 \cdot 130$, compute the least prime number that is NOT a factor of N.

2) Two machines work simultaneously to do a job. Each machine works at its own constant rate. They finish 9 hours sooner than if the first machine had done the job alone. They finish 16 hours sooner than if the second machine had done the job alone. Compute the number of hours it took the two machines to work simultaneously to do the job.

Time: 10 minutes

3) A conqueror was born and died on January 1st in years that were the perfect squares of consecutive positive integers. If she died on her birthday at the age of 55, compute the year of her birth.

4) In rectangle $ABCD$, $AB = 5$ and $BC = 6$. Diagonal $\overline{BD}$ is extended its own length through point D to point E, and $\overline{AE}$ is drawn. Compute AE.

Time: 10 minutes

5) Compute the sum of the real roots of $x^2 + \frac{1}{x^2} - 7x - \frac{7}{x} + 14 = 0$.

6) A fair two-sided coin is tossed until 4 heads occur. The probability that this takes exactly 6 tosses with heads appearing on the fifth and sixth tosses is $\frac{p}{q}$. If p and q are relatively prime (their greatest common factor is 1), compute $p + q$.

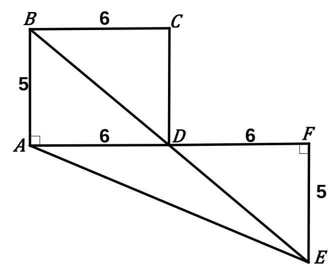
__Solutions for Contest #1__

1) Consider the primes starting with 2. Since 2 is a factor of 20, it is a factor of N. Similarly, 3 is a factor of 30, 5 is a factor of 50, 7 is a factor of 70, 11 is a factor of 110, and 13 is a factor of 130. The next prime is 17 and it is not a factor of any of N's given factors. Thus, the least prime is **17.**

2) Suppose it takes the first machine A hours to do the job alone. Suppose it takes the second machine B hours to do the job alone. Suppose it takes h hours for the two machines to finish the job simultaneously. Then, $\left(\frac{1}{A}+\frac{1}{B}\right)h = \frac{1}{A}(h+9) = \frac{1}{B}(h+16)$. Distribute and subtract to get $\frac{h}{B} = \frac{9}{A}$ and $\frac{h}{A} = \frac{16}{B} \rightarrow \frac{A}{B} = \frac{9}{h}$ and $\frac{A}{B} = \frac{h}{16} \rightarrow h^2 = 144 \rightarrow$ $h = 12$. It takes the machines **12** hours simultaneously to do the job.

3) If x^2 is the year of her birth, then $(x+1)^2 - x^2 = 55 \longrightarrow 2x+1 = 55 \longrightarrow x = 27$. Thus, $x^2 = \mathbf{729}$.

4) From point E, draw a line perpendicular to $\overleftrightarrow{AD}$ which must be extended through point D. The perpendicular line intersects $\overleftrightarrow{AD}$ in point F. By AAS, $\Delta ABD \cong \Delta FED$. So, $EF = 5, DF = 6$, and $AF = 12$. Apply the Pythagorean triple 5-12-13 in right ΔAFE to yield $AE = \mathbf{13}$.



5) Note that $\left(x+\frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 \rightarrow x^2 + \frac{1}{x^2} = \left(x+\frac{1}{x}\right)^2 - 2$. Re-write the given equation as $\left(x+\frac{1}{x}\right)^2 - 2 - 7\left(x+\frac{1}{x}\right) + 14 = 0$. Let $a = x + \frac{1}{x} \rightarrow a^2 - 7a + 12 = 0 \rightarrow a = 3$ or $4 \rightarrow$ $x + \frac{1}{x} = 3$ or $4 \rightarrow x^2 - 3x + 1 = 0$ or $x^2 - 4x + 1 = 0$. The roots of each of these quadratic equations are real. The sum of the roots of these equations are 3 and 4 respectively. The required sum is **7.**

6) Since the last two tosses are heads, exactly two of the first four tosses are heads. The probability of exactly 2 heads in 4 tosses is ${}_4C_2\left(\frac{1}{2}\right)^4 = \frac{3}{8}$. The probability that the last two tosses are both heads is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. So, the required probability is the product of these probabilities, or $\frac{3}{32}$. Thus, the required sum is $3 + 32 = \mathbf{35}$.

Nassau County Interscholastic Mathematics League

Contest #2 Answers must be integers from 0 to 999, inclusive. 2017 – 2018

Calculators are allowed.

Time: 10 minutes

1) Let $A = (1+2+3+4+5+6+7+8+9+10)^{2018}$ and

let $B = (-1-2-3-4-5-6-7-8-9-10)^{2018}$. Compute $A - B$.

2) Leon wants to buy a boat that costs \$100,000. He has only \$60,000 to spend on the boat. The store selling the boat insists on the full price if the boat is sold during the first week. After that, the store will reduce the price of the boat 10% per week based on the previous week's price until the boat is sold. Compute the least number of weeks before the boat is within Leon's budget.

Time: 10 minutes

3) George Greedy went to the racetrack and on his first bet doubled the amount of money he brought to the track. On his second bet, he lost \$800. On his third bet, he bet all his remaining money and doubled it. On his fourth bet, he again lost \$800. On his fifth bet, he again bet all his remaining money and doubled it. On his sixth and final bet, he lost \$800 again and was broke. Compute the number of dollars George had when he started.

4) In parallelogram $WXYZ, WX = 8, WZ = 3$, and $m\measuredangle ZWX = 30$ degrees. A circle whose center is point W contains point Z, and a circle whose center is point Z contains point W. The circles intersect at points P and Q. Point P is closer to point X than point Q is. If $PY = \sqrt{k}$, compute k.

Time: 10 minutes

5) Compute the sum of the roots of $|x-5|^2 - 6|x-5| = -8$.

6) Point C is the intersection of the three medians (centroid) of ΔPQR. A line through point C and parallel to $\overline{PQ}$ intersects $\overline{PR}$ in point A and $\overline{QR}$ in point B. If the area of ΔRAB is 80, compute the area of trapezoid $ABQP$.

__Solutions for Contest #2__

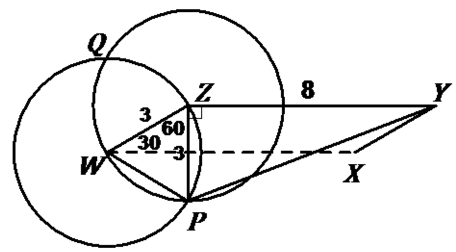
1) The given expression may be re-written as $55^{2018} - (-55)^{2018} = \mathbf{0}$.

2) The price of the boat after n weeks is $\$100000(0.9)^n$, For the boat to be within the budget, $10000(0.9)^n \leq 60000 \rightarrow (0.9)^n \leq 0.6$. A calculator can solve this inequality. Else, $9^n \leq 6 \cdot 10^{n-1}$. Trial and error yields $9^5 = 59049 < 6 \cdot 10^4 = 60000$. Therefore the required number of weeks is **5.**

3) If George started with x dollars, then $2[2(2x - 800) - 800] - 800 = 0 \rightarrow x = \mathbf{700}$.

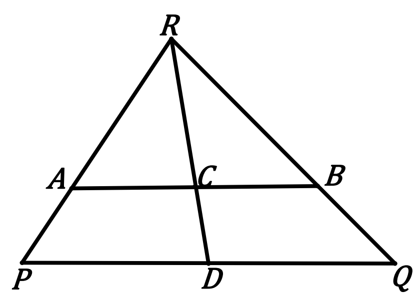
Alternatively, work backwards from 0: $0 \rightarrow 800 \rightarrow 400 \rightarrow 1200 \rightarrow 600 \rightarrow 1400 \rightarrow \mathbf{700.}$

4) The two given circles each have a radius of length 3, so ΔWZP is equilateral, and $m\measuredangle WZP = 60°$. Since $m\measuredangle ZWX = 30°$ and $WXYZ$ is a parallelogram, $m\measuredangle WZY = 150°$. So, $\measuredangle PZY$ is a right angle. By the Pythagorean Theorem in ΔPZY, $PY = \sqrt{3^2 + 8^2} = \sqrt{73}$. Thus, $k = \mathbf{73}$.



5) Re-write the given equation as $|x-5|^2 - 6|x-5| + 8 = 0$. If $a = |x-5|$, then $a^2 - 6a + 8 = 0 \longrightarrow (a-4)(a-2) = 0 \longrightarrow |x-5| = 4$ or $|x-5| = 2$. The solution set is $\{1,9,7,3\}$. The sum of these roots is **20.**

6) Since $\overline{AB} \parallel \overline{PQ}$, $\Delta RAB \sim \Delta RPQ$. If C is the centroid of ΔPQR, then $\overline{RC}$ and $\overline{RD}$ are corresponding medians of the similar triangles. Since a centroid divides a triangle's median into segments whose ratio is $2:1$, $\frac{RC}{RD} = \frac{2}{3} \longrightarrow \frac{[RAB]}{[RPQ]} = \frac{4}{9} \longrightarrow$ $\frac{80}{[RPQ]} = \frac{4}{9} \longrightarrow [RPQ] = 180$. Thus, $[ABQP] = 180 - 80 = \mathbf{100}$.
Note: The symbol $[RPQ]$ means the area of polygon RPQ.



Nassau County Interscholastic Mathematics League

Contest #3 Answers must be integers from 0 to 999, inclusive. 2017 – 2018

No calculators are allowed.

Time: 10 minutes

1) Compute the sum of the prime factors of 399.

2) Two candles that appear to be identical are lit simultaneously. One burns completely uniformly in 3 hours and the other burns completely uniformly in 2 hours. Compute the number of **minutes** it will take for one candle to be exactly twice as long as the other.

Time: 10 minutes

3) If Mr. Skinflint pays for a one-dollar item with 48 coins and receives no change, compute the maximum number of nickels he could use for the purchase.

4) The solution set of $\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$ is $\{x\colon A \le x \le B\}$. Compute $A+B$.

Time: 10 minutes

5) The number 555,555 is expressed as a product of two positive three-digit numbers, T and D. If $T < D$, compute T.

6) In parallelogram $ABCD$, $AB = 13$, $AD = 21$, angle A is acute, and the length of altitude $\overline{BP}$ to side $\overline{AD}$ is 12. The semicircles drawn whose diameters are $\overline{AB}$ and $\overline{AD}$ intersect in points A and X. If $AX = \frac{p}{q}$ is expressed in simplest form, compute $p+q$.

Solutions for Contest #3

1) Since $399 = 400 - 1 = 20^2 - 1^2 = (20+1)(20-1) = 21 \cdot 19 = 3 \cdot 7 \cdot 19$, the required sum is **29.**

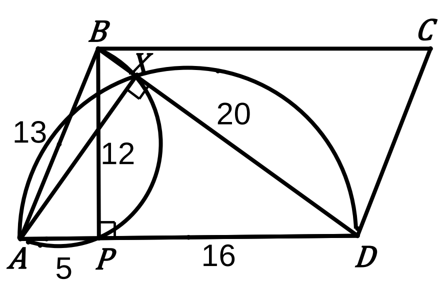
2) Each candle has an initial height of h inches and burns for t minutes until one candle is twice as high as the other. The flame moves down the first candle at a rate of $\frac{h}{3}$ inches per hour and the flame moves down the second candle at a rate of $\frac{h}{2}$ inches per hour. In order for the height of the slower burning candle to be twice the height of the faster burning candle, $h - \frac{h}{3}t = 2\left(h - \frac{h}{2}t\right) \rightarrow 1 - \frac{t}{3} = 2 - t \rightarrow \frac{2t}{3} = 1 \rightarrow t = 1.5$ hours or **90** minutes.

3) To maximize the number of nickels, Mr. Skinflint should not use any dimes or quarters. Otherwise he could use two nickels for each dime and five nickels for each quarter. Suppose he uses N nickels and P pennies. Then $N + P = 48$ and $5N + P = 100$. So, $4N = 52$ and $N =$ **13.**

4) Let $y = \sqrt{x-1}$. Then $y^2 = x - 1$. The original equation becomes $\sqrt{y^2 - 4y + 4} + \sqrt{y^2 - 6y + 9} = 1 \rightarrow \sqrt{(y-2)^2} + \sqrt{(y-3)^2} = 1 \rightarrow |y-2| + |y-3| = 1$. If $y \geq 3$, $y - 2 + y - 3 = 1 \rightarrow y = 3$. If $2 < y < 3$, $y - 2 + 3 - y = 1 \rightarrow 1 = 1$. If $y \leq 2$, $2 - y + 3 - y = 1 \rightarrow y = 2$. So, $2 \leq y \leq 3 \longrightarrow 2 \leq \sqrt{x-1} \leq 3 \longrightarrow 4 \leq x - 1 \leq 9 \longrightarrow 5 \leq x \leq 10$. The required sum is **15.**

5) Since $555{,}555 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 37$, let $T = 5 \cdot 11 \cdot 13 = 715$, and let $D = 3 \cdot 7 \cdot 37 = 777$. It follows that $555{,}555 = 715 \cdot 777$. So, $T =$ **715.**

6) Because point X is on semicircles AXB and AXD, $\measuredangle AXB$ and $\measuredangle AXD$ are right angles and points X, B, and D are collinear. Applying the Pythagorean triple 5-12-13 in ΔABP, $AP = 5$, so $PD = 16$. Applying the Pythagorean triple 12-16-20 in ΔBPD, $BD = 20$. Since $\Delta BPD \sim \Delta AXD$, $\frac{AX}{AD} = \frac{BP}{BD} \rightarrow \frac{AX}{21} = \frac{12}{20} \rightarrow AX = \frac{63}{5}$. The required sum is **68.**



Nassau County Interscholastic Mathematics League

Contest #4 Answers must be integers from 0 to 999, inclusive. 2017 – 2018

Calculators are allowed.

Time: 10 minutes

1) The product of 2^{2017} and 5^{2018} has 2018 digits. Compute the sum of all of these digits.

2) There are two sets, each of five consecutive integers, such that within each set, the sum of the squares of the three least integers equals the sum of the squares of the two greatest integers. Compute the sum of the ten integers that comprise the two sets.

Time: 10 minutes

3) Compute the least positive integer that leaves a remainder of 10 when it is divided into 90.

4) Circle O has a radius of 9, circle P has a radius of 12, and $OP = 54$. From point O, tangent lines are drawn to circle P at points T and B, and they intersect circle O at points Q and R. Compute QR.

Time: 10 minutes

5) If Ms. Speedy travels from home to the airport at 40 mph, she will arrive one hour late. If she travels on the same route at 60 mph, she will be one hour early. If she travels at x mph, she will be on time. Compute x.

6) When $\frac{1}{x} + \frac{1}{y} + \frac{1}{xy} = \frac{1}{3}$ is solved in integers, and all n ordered pairs in the solution set are listed, compute $\sum_{i=1}^{n} (x_i + y_i)$.

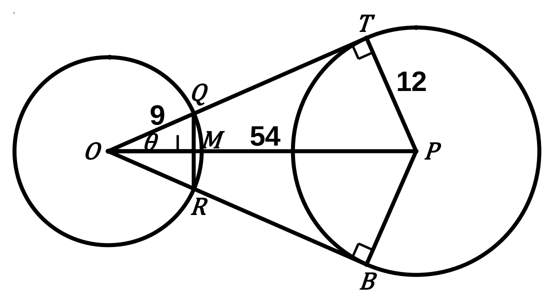
Solutions for Contest #4

1) $2^{2017} \cdot 5^{2018} = 2^{2017} \cdot 5^{2017} \cdot 5 = 5 \cdot 10^{2017}$. This numeral contains a 5 at the front followed by 2017 zeroes. The sum of its digits is **5.**

2) Let the integers be $x-2, x-1, x, x+1, x+2$. Then $(x-2)^2 + (x-1)^2 + x^2 = (x+1)^2 + (x+2)^2 \rightarrow x^2 - 4x + 4 + x^2 - 2x + 1 + x^2 = x^2 + 2x + 1 + x^2 + 4x + 4 \rightarrow 3x^2 - 6x + 5 = 2x^2 + 6x + 5 \longrightarrow -12x + x^2 = 0 \rightarrow x = 0$ or 12. The two sets of integers are $\{-2,-1,0,1,2\}$ and $\{10,11,12,13,14\}$. The sum of the ten integers is **60**.

3) We require $90 = dq + 10$ or $dq = 80$, where q is the quotient and $d > 10$ is the divisor. Solutions, (d, q) to the equation include $(16,5), (20,4), (40,2)$, and $(80,1)$. The least value of $d > 10$ is **16.**

4) Since $\overline{QR}$ is the base of isosceles ΔOQR, the line of centers $\overline{OP} \perp \overline{QR}$ at point M. Let point T be the point of tangency of $\overleftrightarrow{OQ}$ and circle P. Thus, $\overline{OT} \perp \overline{PT}$. Note that $\Delta OMQ \sim \Delta OTP$. As a result, $\frac{OQ}{OP} = \frac{MQ}{TP} \rightarrow \frac{9}{54} = \frac{MQ}{12} \rightarrow MQ = 2$ and $QR = \mathbf{4}$.
Alternatively, let $\theta = m\measuredangle QOM$. In right ΔOTP, $\sin\theta = \frac{12}{54} = \frac{2}{9}$, and in right ΔOMQ, $\sin\theta = \frac{MQ}{9}$. So, $\frac{MQ}{9} = \frac{2}{9} \longrightarrow MQ = 2 \longrightarrow QR = \mathbf{4}$.



5) If she travels from home to the airport at x mph, it will take her t hours to arrive on time. Therefore, $xt = 40(t+1)$ and $xt = 60(t-1)$. So, $40t + 40 = 60t - 60 \rightarrow t = 5$. Substitute that into the first equation to get $5x = 240 \rightarrow x = \mathbf{48}$.

6) When $x \neq 0$ and $y \neq 0$, $\frac{1}{x} + \frac{1}{y} + \frac{1}{xy} = \frac{1}{3} \longrightarrow 3y + 3x + 3 = xy \longrightarrow x + y + 1 = \frac{xy}{3}$.
Since we're looking for integral solutions, 3 divides x or y. Without loss of generality, let $x = 3k$, where k is a non-zero integer. Substitution yields $3k + y + 1 = ky \longrightarrow 3k + 1 = ky - y \longrightarrow y = \frac{3k+1}{k-1} = 3 + \frac{4}{k-1}$. In order for the latter expression to be an integer, the only possible values for k are: $-3, -1, 2, 3$, and 5. These give us the ordered pairs: $(-9, 2), (-3, 1), (6, 7), (9, 5)$ and $(15, 4)$. And, since the given equation has $y = x$ symmetry, we also have $(2, -9), (1, -3), (7, 6), (5, 9)$ and $(4, 15)$ in the solution set. Thus, $n = 10$ and $\sum_{i=1}^{10} (x_i + y_i) = 2[-7 + (-2) + 13 + 14 + 19] = \mathbf{74}$

Nassau County Interscholastic Mathematics League

Contest #5 Answers must be integers from 0 to 999, inclusive. 2017 – 2018

No calculators are allowed.

Time: 10 minutes

1) Compute the only two-digit number that equals twice the sum of its digits.

2) A club of 8 students must send a representative group to the student government council. The representative group must contain at least one club member and may contain as many more club members as the club members desire to send to the council. Compute the number of different representative groups the club may select.

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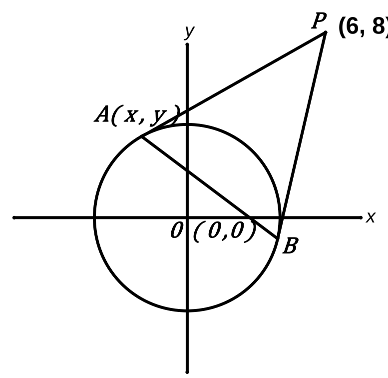
Time: 10 minutes

3) Two points whose coordinates are $(2,9)$ and $(c,6)$ are on a line parallel to the line whose equation is $3x+4y=12$. Compute c.

4) Through point P, with coordinates $(6,8)$ and exterior to circle O, two tangent lines are drawn to the circle. These lines intersect the circle at points A and B. The equation of the circle is $x^2+y^2=16$. The equation of $\overleftrightarrow{AB}$ can be written in the form $ax+by=c$, where a, b, and c are positive constants, and there is no positive factor other than 1 common to all of a, b, and c. Compute $a+b+c$.



Time: 10 minutes

5) In an arithmetic sequence, the first term is 45 and the fifth term is 81. Compute the sum of the second, third, and fourth terms of the arithmetic sequence.

6) If $0 \le x < 4\pi$, compute the number of real roots of

$$\log\left(\sqrt{10}\sin x\right)+\log\left(-1+\sqrt{10}\sin x\right)=\log 6.$$

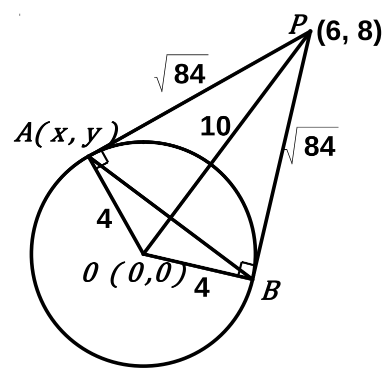
Solutions for Contest #5

1) If we use trial and error, we need not try any odd numbers. We quickly see that **18** works. Alternatively, if the number is $10t + u$, then $10t + u = 2(t + u) \rightarrow u = 8t \rightarrow$ $t = 1$ and $u = 8$. Note that if $t = 2$, $u > 9$, which fails.

2) $\sum_{n=1}^{8} {}_8C_n = 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 = \mathbf{255}.$
Alternatively, the number of subsets for a set containing 8 elements is 2^8. Since we must exclude the empty set, the answer is $2^8 - 1 = \mathbf{255}.$

3) The equation of the given line may be transformed into $y = -\frac{3}{4}x + 3$. The slope of the given line is $-\frac{3}{4}$. Since the two lines are parallel, their slopes are equal. Therefore, $\frac{6-9}{c-2} = -\frac{3}{4} \rightarrow c = \mathbf{6}.$

4) Since $\overline{OA}$ and $\overline{OB}$ are radii of length 4 of circle O, each forms a right angle with the appropriate tangent line to create right triangles OBP and OAP. Since $OP = \sqrt{6^2 + 8^2} = 10$, we use the Pythagorean Theorem to show $PA = PB = \sqrt{10^2 - 4^2} = \sqrt{84}$. If the coordinates of point A are (x, y), then by the distance formula, $(x - 6)^2 + (y - 8)^2 = 84$ and $x^2 + y^2 = 16$. After expanding the binomial squares in the former equation and then subtracting the latter equation from the former, the result is $-12x + 36 - 16y + 64 = 68 \rightarrow -12x - 16y = -32 \rightarrow$ $3x + 4y = 8$. The required sum is $3 + 4 + 8 = \mathbf{15}$. Note: We can assert that this is an equation of $\overleftrightarrow{AB}$ because (x, y) is a point on the circle that is a distance of $\sqrt{84}$ from point P. There are only two such points, A and B.



5) Let the common difference be d. Then, $81 = 45 + 4d \rightarrow d = 9$. Therefore, the second term is 54. The third term is 63. The fourth term is 72. The sum of those three terms is **189.**

6) Rewrite the given equation using the logarithm of a product law:
$\log[(\sqrt{10}\sin x)(-1 + \sqrt{10}\sin x)] = \log 6$. Then, set the arguments equal:
$-\sqrt{10}\sin x + 10\sin^2 x = 6 \rightarrow 10\sin^2 x - \sqrt{10}\sin x - 6 = 0$. By the Quadratic Formula, $\sin x = \frac{3\sqrt{10}}{10}$ or $\frac{-\sqrt{10}}{5}$. We reject $\sin x = \frac{-\sqrt{10}}{5}$ because in the given equation, the argument of the first logarithm, $\sqrt{10}\sin x = -2 < 0$. On the interval $[0, 2\pi]$, $\sin x = \frac{3\sqrt{10}}{10}$ has 2 roots, so on the interval $[0, 4\pi]$, $\sin x = \frac{3\sqrt{10}}{10}$ has **4** roots.

Nassau County Interscholastic Mathematics League

Team Contest   Answers must be integers from 0 to 999, inclusive.   2017 – 2018

Calculators are allowed.

Time: 40 minutes

1) If $10^{51} - 1$ is written as an integer, in standard form, compute the sum of this integer's digits.

2) In a target shooting contest, each shooter hits a whole number of targets. The winner hit twice as many targets as the second place finisher, three times as many as the third place finisher and four times as many as the fourth place finisher. In all, these four contestants hit fewer than 60 targets. Compute the greatest possible number of targets hit by the winner.

3) Two standard dice are tossed together once. The probability that the sum of the numbers shown on the dice is a prime number is $\frac{p}{q}$, expressed in simplest form. Compute $p + q$.

4) A cube whose edge-length is 1 is cut from a corner of a cube whose edge-length is 2. Compute the total surface area of the resulting solid figure.

5) Define the function $f(x)$ to be the area of a semicircle whose diameter has length x. If $f(12) + f(16) = f(a)$, compute a.

6) Compute the sum of all real values of x for which $(x-3)^4 + (x-5)^4 = 16$.

7) The sum of the imaginary roots of $x^3 + 3x^2 + 3x = 1$ is expressible in simplest form as $a - \sqrt[c]{b}$. Compute $(abc)^2$.

8) The lengths of two sides of a triangle and the bisector of their included angle are 12 inches, 15 inches, and 10 inches, respectively. Compute, in inches, the length of the third side of the triangle.

9) From the set $\{1,2,3,4,5\}$, all possible three-element subsets are formed. From each subset, the least element is chosen. If the arithmetic mean of these least elements is expressed in simplest form as $\frac{p}{q}$, compute $p + q$.

10) For all positive integers n, let $h(n)$ denote the sum of the n-th powers of the roots of $x^2 = 4x - 1$. Compute the greatest value of $h(n)$ that is less than 1000.

Solutions for Team Contest

1) For any positive integer n, when 10^n is written as an integer, the first digit on the left is 1 and the other digits are each 0. So, when 10^{51} is written as an integer, it has a 1 on the left followed by 51 zeroes. So , when $10^{51} - 1$ is written as an integer in standard form, it has 51 9's. The sum of its digits is $51 \cdot 9 = \mathbf{459}$.

2) Let the number of targets hit by the winner, the second, the third, and the fourth place finishers be $12x, 6x, 4x$, and $3x$, respectively. So, $12x + 6x + 4x + 3x < 60 \longrightarrow 25x < 60$. The largest integer value of x is 2, so $12x = \mathbf{24}$.

3) $P(\text{prime}) = P(2 \text{ or } 3 \text{ or } 5 \text{ or } 7 \text{ or } 11) = P(2) + P(3) + P(5) + P(7) + P(11) =$

$\frac{1}{36} + \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{2}{36} = \frac{15}{36} = \frac{5}{12}$. The required sum is **17.**

4) In the original cube, after the smaller cube has been cut out, three faces that share a common vertex each have an area of 3, and three additional faces each have an area of 4. In the cut-out cube, three faces each have an area of 1. These areas are part of the new solid figure. So, the sum of all of these areas is the same as the surface area of the original cube: $9 + 12 + 3 = \mathbf{24}$.

5) If the diameter of the semicircle is x, the domain of f is $x > 0$.

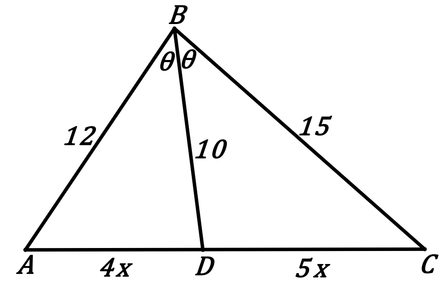
$f(x) = \frac{\pi\left(\frac{x}{2}\right)^2}{2} = \frac{\pi x^2}{8} \longrightarrow f(12) = \frac{144\pi}{8} = 18\pi$, and $f(16) = \frac{256\pi}{8} = 32\pi$.

So, $18\pi + 32\pi = \frac{\pi a^2}{8} \longrightarrow 50\pi = \frac{\pi a^2}{8} \longrightarrow a^2 = 400 \longrightarrow a = \mathbf{20}$.

6) Let $y = x - 4$ and re-write the original equation as $(y+1)^4 + (y-1)^4 = 16 \rightarrow$ $y^4 + 4y^3 + 6y^2 + 4y + 1 + y^4 - 4y^3 + 6y^2 - 4y + 1 = 16 \rightarrow 2y^4 + 12y^2 + 2 = 16 \rightarrow$ $y^4 + 6y^2 - 7 = 0 \rightarrow (y^2 - 1)(y^2 + 7) = 0$. The only real solutions for y are ± 1. So, if $x - 4 = \pm 1$, the real values of x for which $(x-3)^4 + (x-5)^4 = 16$ are 5 and 3. The required sum is **8.**

7) Transform the given equation into $x^3 + 3x^2 + 3x + 1 = 2 \rightarrow (x+1)^3 = 2 \rightarrow$ $x = -1 + \sqrt[3]{2}$, which is real. The sum of all of the roots of the equation is -3. So, the sum of the imaginary roots is $-2 - \sqrt[3]{2}$. Thus, $a = -2$, $b = 2$, $c = 3$, and $(abc)^2 = \mathbf{144}$.

8) Without loss of generality, in ΔABC, point D is on $\overline{ADC}$, where $\overline{BD}$ bisects $\measuredangle ABC$. Let $m\measuredangle ABD = m\measuredangle DBC = \theta$ and using the angle bisector in a triangle theorem, $\frac{12}{15} = \frac{4}{5} = \frac{AD}{DC}$.



Let $AD = 4x$ and let $DC = 5x$. Use the Law of Cosines in the two smaller triangles: $(4x)^2 = 12^2 + 10^2 - 2 \cdot 12 \cdot 10 \cos\theta$ and $(5x)^2 = 15^2 + 10^2 - 2 \cdot 15 \cdot 10 \cos\theta \rightarrow$ $16x^2 = 244 - 240\cos\theta$ and $25x^2 = 325 - 300\cos\theta$. Solve each for $\cos\theta$ and set the resulting expressions equal: $\frac{244-16x^2}{240} = \frac{325-25x^2}{300} \longrightarrow \frac{61-4x^2}{60} = \frac{13-x^2}{12} \longrightarrow$ $732 - 48x^2 = 780 - 60x^2 \longrightarrow 12x^2 = 48 \longrightarrow x = 2$. Thus, $AC = 9x = \mathbf{18}$. Alternatively, after determining the segments $4x$ and $5x$, and the first Law of Cosines equation, apply the Law of Sines: $\frac{4x}{\sin\theta} = \frac{10}{\sin A}$ and $\frac{9x}{\sin 2\theta} = \frac{15}{\sin A}$. Replace $\sin 2\theta$ with $2\sin\theta\cos\theta$ and solve for $\cos\theta = \frac{3}{4}$. Then substitute into the Law of Cosines to get $9x = \mathbf{18}$.

9) There are ${}_5C_3 = 10$ subsets. From these, ${}_4C_2 = 6$ subsets have 1 as their least element, ${}_3C_2 = 3$ subsets have 2 as their least element, and ${}_2C_2 = 1$ subset has 3 as its least element. The mean of these least elements is $\frac{6\cdot 1 + 3\cdot 2 + 1\cdot 3}{10} = \frac{15}{10} = \frac{3}{2}$.
The required sum is **5.**

10) Let a and b be the roots of the given equation. Multiply both sides of the given equation by $x^n \rightarrow x^{n+2} = 4x^{n+1} - x^n \rightarrow a^{n+2} = 4a^{n+1} - a^n$ and $b^{n+2} = 4b^{n+1} - b^n$. Add the latter two equations to yield $h(n+2) = 4h(n+1) - h(n)$ for all positive integers n. Use this recursion together with the fact that $h(0) = 2$ and $h(1) = 4$ to discover that $h(2) = 14$, $h(3) = 52$, $h(4) = 194$, $h(5) = 724$, and $h(6) = 2702$. The required value of the function is **724.**
Alternatively, by the Quadratic Formula, the roots of the given quadratic equation are $2 \pm \sqrt{3}$. Use a calculator to evaluate $\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n$ for integral values of $n \geq 1$ to determine that $\left(2+\sqrt{3}\right)^5 + \left(2-\sqrt{3}\right)^5 = \mathbf{724}$ and that $\left(2+\sqrt{3}\right)^6 + \left(2-\sqrt{3}\right)^6 >$ 1000.