Nassau County Interscholastic Mathematics League

Team Contest   Answers must be integers from 0 to 999, inclusive.   2017 – 2018

Calculators are allowed.

Time: 40 minutes

31) If $10^{51} - 1$ is written as an integer, in standard form, compute the sum of this integer's digits.

32) In a target shooting contest, each shooter hits a whole number of targets. The winner hit twice as many targets as the second place finisher, three times as many as the third place finisher and four times as many as the fourth place finisher. In all, these four contestants hit fewer than 60 targets. Compute the greatest possible number of targets hit by the winner.

33) Two standard dice are tossed together once. The probability that the sum of the numbers shown on the dice is a prime number is $\frac{p}{q}$, expressed in simplest form. Compute $p + q$.

34) A cube whose edge-length is 1 is cut from a corner of a cube whose edge-length is 2. Compute the total surface area of the resulting solid figure.

35) Define the function $f(x)$ to be the area of a semicircle whose diameter has length x. If $f(12) + f(16) = f(a)$, compute a.

36) Compute the sum of all real values of x for which $(x-3)^4 + (x-5)^4 = 16$.

37) The sum of the imaginary roots of $x^3 + 3x^2 + 3x = 1$ is expressible in simplest form as $a - \sqrt[c]{b}$. Compute $(abc)^2$.

38) The lengths of two sides of a triangle and the bisector of their included angle are 12 inches, 15 inches, and 10 inches, respectively. Compute, in inches, the length of the third side of the triangle.

39) From the set $\{1,2,3,4,5\}$, all possible three-element subsets are formed. From each subset, the least element is chosen. If the arithmetic mean of these least elements is expressed in simplest form as $\frac{p}{q}$, compute $p + q$.

40) For all positive integers n, let $h(n)$ denote the sum of the n-th powers of the roots of $x^2 = 4x - 1$. Compute the greatest value of $h(n)$ that is less than 1000.

Solutions for Team Contest

31) For any positive integer n, when 10^n is written as an integer, the first digit on the left is 1 and the other digits are each 0. So, when 10^{51} is written as an integer, it has a 1 on the left followed by 51 zeroes. So , when $10^{51} - 1$ is written as an integer in standard form, it has 51 9's. The sum of its digits is $51 \cdot 9 = \mathbf{459}$.

32) Let the number of targets hit by the winner, the second, the third, and the fourth place finishers be $12x, 6x, 4x$, and $3x$, respectively. So, $12x + 6x + 4x + 3x < 60 \longrightarrow 25x < 60$. The largest integer value of x is 2, so $12x = \mathbf{24}$.

33) $P(\text{prime}) = P(2 \text{ or } 3 \text{ or } 5 \text{ or } 7 \text{ or } 11) = P(2) + P(3) + P(5) + P(7) + P(11) =$

$\frac{1}{36} + \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{2}{36} = \frac{15}{36} = \frac{5}{12}$. The required sum is **17.**

34) In the original cube, after the smaller cube has been cut out, three faces that share a common vertex each have an area of 3, and three additional faces each have an area of 4. In the cut-out cube, three faces each have an area of 1. These areas are part of the new solid figure. So, the sum of all of these areas is the same as the surface area of the original cube: $9 + 12 + 3 = \mathbf{24}$.

35) If the diameter of the semicircle is x, the domain of f is $x > 0$.

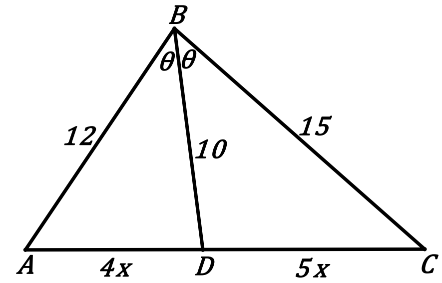
$f(x) = \frac{\pi\left(\frac{x}{2}\right)^2}{2} = \frac{\pi x^2}{8} \longrightarrow f(12) = \frac{144\pi}{8} = 18\pi$, and $f(16) = \frac{256\pi}{8} = 32\pi$.

So, $18\pi + 32\pi = \frac{\pi a^2}{8} \longrightarrow 50\pi = \frac{\pi a^2}{8} \longrightarrow a^2 = 400 \longrightarrow a = \mathbf{20}$.

36) Let $y = x - 4$ and re-write the original equation as $(y+1)^4 + (y-1)^4 = 16 \rightarrow$ $y^4 + 4y^3 + 6y^2 + 4y + 1 + y^4 - 4y^3 + 6y^2 - 4y + 1 = 16 \rightarrow 2y^4 + 12y^2 + 2 = 16 \rightarrow$ $y^4 + 6y^2 - 7 = 0 \rightarrow (y^2 - 1)(y^2 + 7) = 0$. The only real solutions for y are ± 1. So, if $x - 4 = \pm 1$, the real values of x for which $(x-3)^4 + (x-5)^4 = 16$ are 5 and 3. The required sum is **8.**

37) Transform the given equation into $x^3 + 3x^2 + 3x + 1 = 2 \rightarrow (x+1)^3 = 2 \rightarrow$ $x = -1 + \sqrt[3]{2}$, which is real. The sum of all of the roots of the equation is -3. So, the sum of the imaginary roots is $-2 - \sqrt[3]{2}$. Thus, $a = -2$, $b = 2$, $c = 3$, and $(abc)^2 = \mathbf{144}$.

38) Without loss of generality, in ΔABC, point D is on $\overline{ADC}$, where $\overline{BD}$ bisects $\measuredangle ABC$. Let $m\measuredangle ABD = m\measuredangle DBC = \theta$ and using the angle bisector in a triangle theorem, $\frac{12}{15} = \frac{4}{5} = \frac{AD}{DC}$.



Let $AD = 4x$ and let $DC = 5x$. Use the Law of Cosines in the two smaller triangles: $(4x)^2 = 12^2 + 10^2 - 2 \cdot 12 \cdot 10 \cos\theta$ and $(5x)^2 = 15^2 + 10^2 - 2 \cdot 15 \cdot 10 \cos\theta \rightarrow$ $16x^2 = 244 - 240\cos\theta$ and $25x^2 = 325 - 300\cos\theta$. Solve each for $\cos\theta$ and set the resulting expressions equal: $\frac{244-16x^2}{240} = \frac{325-25x^2}{300} \longrightarrow \frac{61-4x^2}{60} = \frac{13-x^2}{12} \longrightarrow$ $732 - 48x^2 = 780 - 60x^2 \longrightarrow 12x^2 = 48 \longrightarrow x = 2$. Thus, $AC = 9x = \mathbf{18}$. Alternatively, after determining the segments $4x$ and $5x$, and the first Law of Cosines equation, apply the Law of Sines: $\frac{4x}{\sin\theta} = \frac{10}{\sin A}$ and $\frac{9x}{\sin 2\theta} = \frac{15}{\sin A}$. Replace $\sin 2\theta$ with $2\sin\theta\cos\theta$ and solve for $\cos\theta = \frac{3}{4}$. Then substitute into the Law of Cosines to get $9x = \mathbf{18}$.

39) There are ${}_5C_3 = 10$ subsets. From these, ${}_4C_2 = 6$ subsets have 1 as their least element, ${}_3C_2 = 3$ subsets have 2 as their least element, and ${}_2C_2 = 1$ subset has 3 as its least element. The mean of these least elements is $\frac{6 \cdot 1 + 3 \cdot 2 + 1 \cdot 3}{10} = \frac{15}{10} = \frac{3}{2}$.
The required sum is **5.**

40) Let a and b be the roots of the given equation. Multiply both sides of the given equation by $x^n \rightarrow x^{n+2} = 4x^{n+1} - x^n \rightarrow a^{n+2} = 4a^{n+1} - a^n$ and $b^{n+2} = 4b^{n+1} - b^n$. Add the latter two equations to yield $h(n+2) = 4h(n+1) - h(n)$ for all positive integers n. Use this recursion together with the fact that $h(0) = 2$ and $h(1) = 4$ to discover that $h(2) = 14$, $h(3) = 52$, $h(4) = 194$, $h(5) = 724$, and $h(6) = 2702$. The required value of the function is **724.**
Alternatively, by the Quadratic Formula, the roots of the given quadratic equation are $2 \pm \sqrt{3}$. Use a calculator to evaluate $\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n$ for integral values of $n \geq 1$ to determine that $\left(2+\sqrt{3}\right)^5 + \left(2-\sqrt{3}\right)^5 = \mathbf{724}$ and that $\left(2+\sqrt{3}\right)^6 + \left(2-\sqrt{3}\right)^6 > 1000$.