Nassau County Interscholastic Mathematics League

Contest #4 Answers must be integers from 0 to 999, inclusive. 2017 – 2018

Calculators are allowed.

**Time: 10 minutes** **Name: ______________________________**

19) The product of $2^{2017}$ and $5^{2018}$ has 2018 digits. Compute the sum of all of these digits.

20) There are two sets, each of five consecutive integers, such that within each set, the sum of the squares of the three least integers equals the sum of the squares of the two greatest integers. Compute the sum of the ten integers that comprise the two sets.

19. 

20. 

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21) Compute the least positive integer that leaves a remainder of 10 when it is divided into 90.

22) Circle $O$ has a radius of 9, circle $P$ has a radius of 12, and $OP = 54$. From point $O$, tangent lines are drawn to circle $P$ at points $T$ and $B$, and they intersect circle $O$ at points $Q$ and $R$. Compute $QR$.

21. [ ]

22. [ ]

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23) If Ms. Speedy travels from home to the airport at 40 mph, she will arrive one hour late. If she travels on the same route at 60 mph, she will be one hour early. If she travels at $x$ mph, she will be on time. Compute $x$.

24) When $\frac{1}{x}+\frac{1}{y}+\frac{1}{xy}=\frac{1}{3}$ is solved in integers, and all $n$ ordered pairs in the solution set are listed, compute $\sum_{i=1}^{n}(x_i+y_i)$.

23. ______

24. ______

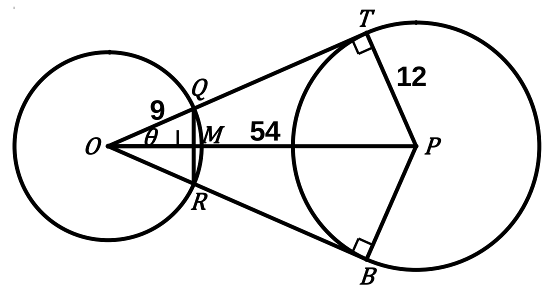
**Solutions for Contest #4**

19) $2^{2017} \cdot 5^{2018} = 2^{2017} \cdot 5^{2017} \cdot 5 = 5 \cdot 10^{2017}$. This numeral contains a 5 at the front followed by 2017 zeroes. The sum of its digits is **5.**

20) Let the integers be $x-2, x-1, x, x+1, x+2$. Then $(x-2)^2 + (x-1)^2 + x^2 = (x+1)^2 + (x+2)^2 \rightarrow x^2 - 4x + 4 + x^2 - 2x + 1 + x^2 = x^2 + 2x + 1 + x^2 + 4x + 4 \rightarrow 3x^2 - 6x + 5 = 2x^2 + 6x + 5 \longrightarrow -12x + x^2 = 0 \rightarrow x = 0$ or 12. The two sets of integers are $\{-2,-1,0,1,2\}$ and $\{10,11,12,13,14\}$. The sum of the ten integers is **60**.

21) We require $90 = dq + 10$ or $dq = 80$, where $q$ is the quotient and $d > 10$ is the divisor. Solutions, $(d, q)$ to the equation include $(16,5), (20,4), (40,2)$, and $(80,1)$. The least value of $d > 10$ is **16.**

22) Since $\overline{QR}$ is the base of isosceles $\Delta OQR$, the line of centers $\overline{OP} \perp \overline{QR}$ at point $M$. Let point $T$ be the point of tangency of $\overleftrightarrow{OQ}$ and circle $P$. Thus, $\overline{OT} \perp \overline{PT}$. Note that $\Delta OMQ \sim \Delta OTP$. As a result, $\frac{OQ}{OP} = \frac{MQ}{TP} \rightarrow \frac{9}{54} = \frac{MQ}{12} \rightarrow$ $MQ = 2$ and $QR = \mathbf{4}$.
Alternatively, let $\theta = m\measuredangle QOM$. In right $\Delta OTP$, $\sin\theta = \frac{12}{54} = \frac{2}{9}$, and in right $\Delta OMQ$, $\sin\theta = \frac{MQ}{9}$ . So, $\frac{MQ}{9} = \frac{2}{9} \longrightarrow MQ = 2 \longrightarrow QR = \mathbf{4}$.




23) If she travels from home to the airport at $x$ mph, it will take her $t$ hours to arrive on time. Therefore, $xt = 40(t+1)$ and $xt = 60(t-1)$. So, $40t + 40 = 60t - 60 \rightarrow t = 5$. Substitute that into the first equation to get $5x = 240 \rightarrow x = \mathbf{48}$.

24) When $x \neq 0$ and $y \neq 0$, $\frac{1}{x} + \frac{1}{y} + \frac{1}{xy} = \frac{1}{3} \longrightarrow 3y + 3x + 3 = xy \longrightarrow x + y + 1 = \frac{xy}{3}$.
Since we're looking for integral solutions, 3 divides $x$ or $y$. Without loss of generality, let $x = 3k$, where $k$ is a non-zero integer. Substitution yields $3k + y + 1 = ky \longrightarrow 3k + 1 = ky - y \longrightarrow y = \frac{3k+1}{k-1} = 3 + \frac{4}{k-1}$. In order for the latter expression to be an integer, the only possible values for $k$ are: $-3, -1, 2, 3$, and $5$. These give us the ordered pairs: $(-9, 2), (-3, 1), (6, 7), (9, 5)$ and $(15, 4)$. And, since the given equation has $y = x$ symmetry, we also have $(2, -9), (1, -3), (7, 6), (5, 9)$ and $(4, 15)$ in the solution set. Thus, $n = 10$ and $\sum_{i=1}^{10} (x_i + y_i) = 2[-7 + (-2) + 13 + 14 + 19] = \mathbf{74}$