Nassau County Interscholastic Mathematics League

Contest #1 Answers must be integers from 0 to 999, inclusive. 2017 – 2018

No calculators are allowed.

Time: 10 minutes

1) If $N = 20 \cdot 30 \cdot 50 \cdot 70 \cdot 110 \cdot 130$, compute the least prime number that is NOT a factor of N.

2) Two machines work simultaneously to do a job. Each machine works at its own constant rate. They finish 9 hours sooner than if the first machine had done the job alone. They finish 16 hours sooner than if the second machine had done the job alone. Compute the number of hours it took the two machines to work simultaneously to do the job.

1. [ ]

2. [ ]

Time: 10 minutes

3) A conqueror was born and died on January 1st in years that were the perfect squares of consecutive positive integers. If she died on her birthday at the age of 55, compute the year of her birth.

4) In rectangle $ABCD$, $AB = 5$ and $BC = 6$. Diagonal $\overline{BD}$ is extended its own length through point D to point E, and $\overline{AE}$ is drawn. Compute AE.

3. ☐

4. ☐

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5) Compute the sum of the real roots of $x^2+\frac{1}{x^2}-7x-\frac{7}{x}+14=0$.

6) A fair two-sided coin is tossed until 4 heads occur. The probability that this takes exactly 6 tosses with heads appearing on the fifth and sixth tosses is $\frac{p}{q}$. If p and q are relatively prime (their greatest common factor is 1), compute $p+q$.

5. [ ]

6. [ ]

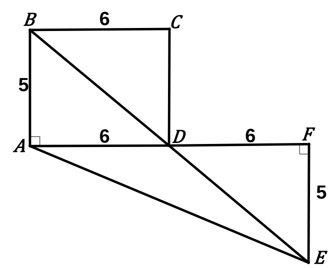
Solutions for Contest #1

1) Consider the primes starting with 2. Since 2 is a factor of 20, it is a factor of N. Similarly, 3 is a factor of 30, 5 is a factor of 50, 7 is a factor of 70, 11 is a factor of 110, and 13 is a factor of 130. The next prime is 17 and it is not a factor of any of N's given factors. Thus, the least prime is **17.**

2) Suppose it takes the first machine A hours to do the job alone. Suppose it takes the second machine B hours to do the job alone. Suppose it takes h hours for the two machines to finish the job simultaneously. Then, $\left(\frac{1}{A}+\frac{1}{B}\right)h=\frac{1}{A}(h+9)=\frac{1}{B}(h+16)$. Distribute and subtract to get $\frac{h}{B}=\frac{9}{A}$ and $\frac{h}{A}=\frac{16}{B}\rightarrow\frac{A}{B}=\frac{9}{h}$ and $\frac{A}{B}=\frac{h}{16}\rightarrow h^2=144\rightarrow$ $h=12$. It takes the machines **12** hours simultaneously to do the job.

3) If x^2 is the year of her birth, then $(x+1)^2-x^2=55\longrightarrow 2x+1=55\longrightarrow x=27$. Thus, $x^2=\mathbf{729}$.

4) From point E, draw a line perpendicular to $\overleftrightarrow{AD}$ which must be extended through point D. The perpendicular line intersects $\overleftrightarrow{AD}$ in point F. By AAS, $\Delta ABD\cong\Delta FED$. So, $EF=5, DF=6$, and $AF=12$. Apply the Pythagorean triple 5-12-13 in right ΔAFE to yield $AE=\mathbf{13}$.



5) Note that $\left(x+\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}+2\rightarrow x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2$. Re-write the given equation as $\left(x+\frac{1}{x}\right)^2-2-7\left(x+\frac{1}{x}\right)+14=0$. Let $a=x+\frac{1}{x}\rightarrow a^2-7a+12=0\rightarrow a=3$ or $4\rightarrow$ $x+\frac{1}{x}=3$ or $4\rightarrow x^2-3x+1=0$ or $x^2-4x+1=0$. The roots of each of these quadratic equations are real. The sum of the roots of these equations are 3 and 4 respectively. The required sum is **7.**

6) Since the last two tosses are heads, exactly two of the first four tosses are heads. The probability of exactly 2 heads in 4 tosses is ${}_4C_2\left(\frac{1}{2}\right)^4=\frac{3}{8}$. The probability that the last two tosses are both heads is $\left(\frac{1}{2}\right)^2=\frac{1}{4}$. So, the required probability is the product of these probabilities, or $\frac{3}{32}$. Thus, the required sum is $3+32=\mathbf{35}$.