Nassau County Interscholastic Mathematics League

Contest #1 Answers must be integers from 0 to 999, inclusive. 2015 – 2016

No calculators are allowed.

Time: 10 minutes

1) Compute the units digit of 7^{2016}.

2) The length of a side of square $PQRS$ is 4. Four circles, each with radius 2 are drawn with centers $P, Q, R,$ and S. A smaller circle is externally tangent to each of these four circles. If the radius of the smaller circle in simplest form is $a\sqrt{b} - c$, compute $a + b + c$.

Time: 10 minutes

3) Compute a number which is 1000 more than its additive inverse.

4) The angle bisector to the hypotenuse of a right triangle separates the hypotenuse into two line segments whose lengths are 3 and 6. If the area of the given right triangle, in simplest form, is $\frac{p}{q}$, compute $p + q$.

Time: 10 minutes

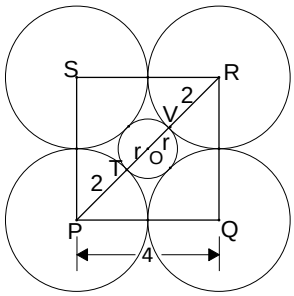
5) If it is given that $x \,\Delta\, y = 2x + 3y$, compute the value of $\{5 \,\Delta\, [(3 \,\Delta\, 2) \,\Delta\, 4]\} \,\Delta\, 6$.

6) A bag contains 3 red marbles, 4 white marbles, and 5 blue marbles. The marbles are indistinguishable except for color. Two marbles are chosen randomly without replacement. If the probability, in simplest form, that they are different colors is $\frac{p}{q}$, compute $p + q$.

Solutions for Contest #1

1) By computing $7^0 = 1,\ 7^1 = 7,\ 7^2 = 49,\ 7^3 = 343,\ 7^4 = 2401,$ etc., we see that the sequence of units digits, 1, 7, 9, 3, 1, 7, 9, 3, … repeats in a cycle of 4. Since 2016 divided by 4 has a remainder of zero, the units digit of 7^{2016} is the same as 7^0, or **1.**

2) Draw diagonal $\overline{PR}$. Its midpoint, O, is the center of the smaller circle and $PR = 4\sqrt{2}$. Points T and V are points of tangency between the smaller circle and circles P and R, respectively. So, $PT = VR = 2$. Let the required radius $r = TO = VO$. So, $PR = PT + TO + OV + VR$. Thus, $4\sqrt{2} = 2 + r + r + 2$ and $r = 2\sqrt{2} - 2$. Therefore, $a = b = c = 2$ and the required sum is **6.**



3) If the number is x, then $x = -x + 1000 \longrightarrow 2x = 1000$ and $x = \mathbf{500}$.

4) By the Angle Bisector Theorem of a triangle, the lengths of the legs of the right triangle are in a 3:6 or 1:2 ratio. Let the lengths of the legs be x and $2x$. By the Pythagorean Theorem, $x^2 + (2x)^2 = (6+3)^2 \longrightarrow 5x^2 = 81 \longrightarrow x^2 = \frac{81}{5}$. The area of the right triangle is $\frac{1}{2} \cdot x \cdot 2x = x^2$. Therefore, $p = 81$ and $q = 5$. The required sum is **86.**

5) To evaluate this expression, we start with the innermost grouping symbols, and work our way out to the outermost grouping symbols: So,
$3\ \Delta\ 2 = 12 \longrightarrow 12\ \Delta\ 4 = 36 \longrightarrow 5\ \Delta\ 36 = 118 \longrightarrow 118\ \Delta\ 6 = \mathbf{254}$.

6) The required probability can be computed as the sum of the probabilities of a red and white in either order, a red and a blue in either order, and a white and a blue in either order. This results in $\frac{{}_3C_1 \cdot {}_4C_1 + {}_3C_1 \cdot {}_5C_1 + {}_4C_1 \cdot {}_5C_1}{{}_{12}C_2} = \frac{47}{66}$. Thus, $p = 47, q = 66$, and $p + q = \mathbf{113.}$
Alternatively, the sum of the probabilities can be expressed as:

$$\frac{3}{12} \cdot \frac{4}{11} + \frac{4}{12} \cdot \frac{3}{11} + \frac{3}{12} \cdot \frac{5}{11} + \frac{5}{12} \cdot \frac{3}{11} + \frac{4}{12} \cdot \frac{5}{11} + \frac{5}{12} \cdot \frac{4}{11} = \frac{24+30+40}{132} = \frac{94}{132} = \frac{47}{66}.$$

So, $47 + 66 = \mathbf{113}$.

Nassau County Interscholastic Mathematics League

Contest #2 Answers must be integers from 0 to 999, inclusive. 2015 – 2016

Calculators are allowed.

Time: 10 minutes

1) The sum of the proper divisors of 18 is 21. Compute the sum of the proper divisors of 150.

2) Compute the remainder when the number of arrangements of 5 students sitting in a row of 9 seats is divided by 1000.

Time: 10 minutes

3) Circle A is entirely interior to circle B. The radius of circle A is 4. The area of the region interior to circle B but exterior to circle A is 180π. Compute the radius of circle B.

4) All three of the lines whose equations are $3x - 4y = 10, 7x - ay = 16$, and $5x - by = 24$ are concurrent in a single point. Compute the numerical value of $a + b$.

Time: 10 minutes

5) Compute the number of integral solutions of $1 \leq \left|x - \frac{1}{2}\right| \leq 6$.

6) A bag contains 4 red marbles, 7 white marbles, and 10 blue marbles. The marbles are indistinguishable except for color. Three marbles are chosen randomly without replacement. If the probability, in simplest form, that at least two of them are the same color is $\frac{p}{q}$, compute pq.

Solutions for Contest #2

1) The divisors of 150 are 1, 150, 2, 75, 3, 50, 5, 30, 6, 25, 10, and 15. The proper divisors of 150 are all of the above except 150. The sum of the proper divisors of 150 is **222.**
Alternatively, use the formula for the sum of the divisors of an integer: If the prime factorization of an integer is $a^k \cdot b^n$, then the sum of the integer's divisors is $(a^0 + a^1 + a^2 + \cdots + a^k)(b^0 + b^1 + b^2 + \cdots + b^n)$. So, for $150 = 2 \cdot 3 \cdot 5^2$, the sum of the divisors is $(2^0 + 2^1)(3^0 + 3^1)(5^0 + 5^1 + 5^2) = (3)(4)(31) = 372$. Since we are asked for the sum of the proper divisors, we must subtract 150 to get $372 - 150 = \mathbf{222}$.

2) Five students can be seated in nine seats in ${}_9P_5 = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15{,}120$ ways. When this number is divided by 1000, the quotient is 15 and the required remainder is **120.**

3) Let the radius of circle B be R. The sum of the area interior to circle A and the area interior to circle B, but exterior to circle A, equals the area of the region interior to circle B. So, $16\pi + 180\pi = \pi R^2$. Thus, $196\pi = \pi R^2 \longrightarrow R^2 = 196$ and $R = \mathbf{14}$.

4) Add the second and third equations to get $12x - (a + b)y = 40$. Multiply the first equation by 4 to get $12x - 16y = 40$. Since the three given lines are all concurrent, then the coefficients of y of these two equations must be equal. Thus, $a + b = 16$.
Alternatively, we can subtract these two equations to get $-(a + b)y + 16y = 0$. Thus, $16y = (a + b)y$ and $a + b = \mathbf{16}$.

5) The given sandwich inequality consists of the <u>intersection</u> of the two inequalities: $\left|x - \frac{1}{2}\right| \geq 1$ **AND** $\left|x - \frac{1}{2}\right| \leq 6$. The first is solved by numbers that are at least 1 unit from $\frac{1}{2}$, that is, $x \leq -\frac{1}{2}$ or $x \geq \frac{3}{2}$. The second is solved by numbers that are at most 6 units from $\frac{1}{2}$, that is, $-\frac{11}{2} \leq x \leq \frac{13}{2}$. Thus, the <u>intersection</u> of these solution sets, in integers, is $\{-5, -4, -3, -2, -1, 2, 3, 4, 5, 6\}$. There are a total of **10** integer solutions.

   Alternatively, we can use the definition of absolute value to write the given inequality as the <u>union</u> of the two inequalities: $1 \leq \frac{1}{2} - x \leq 6$, for $x < \frac{1}{2}$ **OR** $1 \leq x - \frac{1}{2} \leq 6$, for $x \geq \frac{1}{2}$. These yield $-\frac{11}{2} \leq x \leq -\frac{1}{2}$ OR $\frac{3}{2} \leq x \leq \frac{13}{2}$. Thus, the <u>union</u> of these two sets, in integers, is $\{-5, -4, -3, -2, -1, 2, 3, 4, 5, 6\}$, and there are **10** integer solutions.

6) The required probability can be computed as $1 - P(\text{all three are different colors})$. This is $1 - \frac{{}_4C_1 \cdot {}_7C_1 \cdot {}_{10}C_1}{{}_{21}C_3} = 1 - \frac{280}{1330} = \frac{15}{19}$. Thus, $p = 15$, $q = 19$, and $pq = \mathbf{285}$.

Nassau County Interscholastic Mathematics League

Contest #3 Answers must be integers from 0 to 999, inclusive. 2015 – 2016

No calculators are allowed.

Time: 10 minutes

1) On rectangle $ABCD$, point E is the midpoint of $\overline{AD}$ and point F is the midpoint of $\overline{AB}$. If the area of the triangular region EAF is 100, compute the area of the rectangular region $ABCD$.

2) If $x_1 = 1$ and $x_{n+1} = \frac{x_n}{x_n + 1}$, then if, in simplest form, $x_{2015} = \frac{a}{b}$, compute half of the arithmetic mean of a and b.

Time: 10 minutes

3) If $xy = 30$ and $x - y = 13$, compute $|x + y|$.

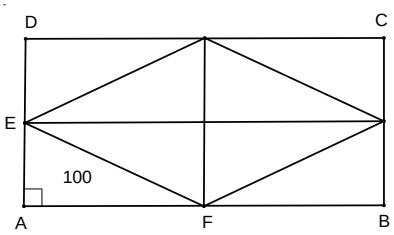
4) On the interval $[0, 3\pi]$, the graph of $y = \cos^2(x) - \sin^2(x) - \frac{1}{2}$ intersects the x-axis multiple times. The sum of the x-intercepts on this interval is $n\pi$. Compute n.

Time: 10 minutes

5) Compute the number of positive integral divisors of 12^{11}.

6) In trapezoid $PQRS$, bases $\overline{PQ}$ and $\overline{RS}$ have lengths of 12 and 4, respectively. Point X is on $\overline{PS}$, point Y is on $\overline{QR}$, and $\overline{XY} \parallel \overline{PQ}$. Trapezoid $PQYX$ is equal in area to trapezoid $SRYX$. If $XY = \sqrt{k}$, compute k.

Solutions for Contest #3



1) Rectangular region $ABCD$ can be tiled so that exactly eight congruent triangular regions fit in the rectangular region. So the required area is **800**. Alternatively, Let $AD = x$ and $AB = y$. Then, area$(\Delta EAF) = \frac{1}{2} \cdot \frac{1}{2} x \cdot \frac{1}{2} y = \frac{1}{8} xy = 100$. So, $xy =$ **800.**

2) Computing the first few iterations, we see that when $x_1 = 1$, we get $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{3}$, $x_4 = \frac{1}{4}$, and so on. So, if $x_k = \frac{1}{k}$, and $x_{k+1} = \frac{x_k}{x_k + 1} = \frac{\frac{1}{k}}{\frac{1}{k}+1} = \frac{1}{k+1}$, then $x_{2015} = \frac{1}{2015}$. Thus, $a = 1$ and $b = 2015$, and half of their arithmetic mean is $\frac{1}{2}\left(\frac{1+2015}{2}\right) = \frac{1}{4}(2016) =$ **504**.

3) Square the second given equation: $x^2 - 2xy + y^2 = 169$. Add $4xy$ to both sides to get $x^2 + 2xy + y^2 = 169 + 4xy$. So, $(x + y)^2 = 169 + 4(30) = 289$. Therefore, $|x + y| =$ **17**. Alternatively, by inspection, $15 \cdot 2 = 30$, and $15 - 2 = 13$. Therefore, $|15 + 2| =$ **17**.

4) Let $y = 0$ and use the identity $\cos(2x) = \cos^2(x) - \sin^2(x)$ to solve $\cos^2(x) - \sin^2(x) = \frac{1}{2}$ $\longrightarrow \cos(2x) = \frac{1}{2} \longrightarrow 2x = \frac{\pi}{3} + 2\pi k$ or $2x = \frac{5\pi}{3} + 2\pi k$, where k is an integer. So, $x = \frac{\pi}{6} + \pi k$ or $x = \frac{5\pi}{6} + \pi k$. On the interval $[0, 3\pi] = \left[0, \frac{18\pi}{6}\right]$, the x-intercepts, when $k = 0, 1, 2$ are: $\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}$. The sum of these x-intercepts is 9π. Thus, $n =$ **9**. Alternatively, we can use $\cos^2(x) = 1 - \sin^2(x)$ or $\sin^2(x) = 1 - \cos^2(x)$ to solve the resulting second-degree trig equation.

5) Since $12^{11} = (2^2 \cdot 3)^{11} = 2^{22} \cdot 3^{11}$, the positive integral divisors of 12^{11} must have only 2's or 3's as their prime factors. That is, each factor of 12^{11} must be written as $2^x \cdot 3^y$, where x and y are integers, and $0 \le x \le 22$ and $0 \le y \le 11$. Thus, there are 23 possible values for x and 12 possible values for y. Therefore, there are $23 \cdot 12 =$ **276** positive integral divisors of 12^{11}. In general, if a number's factorization is $a^x \cdot b^y$, where a and b are prime, and x and y are whole numbers, then the number of integral divisors is $(x + 1)(y + 1)$.

6) Extend $\overline{PS}$ through point S and extend $\overline{QR}$ through point R. Let the point of intersection of the two extensions be point T. Triangles TPQ, TSR, and TXY are thus formed and they are similar to each other. Let the area of ΔTSR be A and let the areas of trapezoid $PQYX$ and trapezoid $SRYX$ each be M. Then, the areas of two similar triangles are to each other as the square of the ratio of their corresponding sides. For $\Delta TSR \sim \Delta TPQ$, $\frac{A}{A+2M} = \left(\frac{4}{12}\right)^2 = \frac{1}{9} \longrightarrow$ $9A = A + 2M \longrightarrow 8A = 2M \longrightarrow 4A = M$. For $\Delta TSR \sim \Delta TXY$, $\frac{A}{A+M} = \left(\frac{4}{\sqrt{k}}\right)^2$. Substitute and square to yield $\frac{A}{5A} = \frac{16}{k}$. Reduce and solve to get $k =$ **80.**

Nassau County Interscholastic Mathematics League

Contest #4 Answers must be integers from 0 to 999, inclusive. 2015 – 2016

Calculators are allowed.

Time: 10 minutes

1) Compute the number of positive integral divisors of 11^{12}.

2) The point whose coordinates are $(10,7)$ is on the graph of $y = f(x)$. The point whose coordinates are (c, d) is on the graph of $y = 3f(x - 5)$. Compute $c + d$.

Time: 10 minutes

3) Any tradesperson working alone can do a certain job in 4 hours. Any apprentice working alone can do the same job in 6 hours. If 2 tradespeople and 3 apprentices work together and do not get in each other's way, then compute the number of hours it will take for all of them to do this job together.

4) The distance between lines l and m, whose equations are $y = 2x$ and $y = 2x + 15$, respectively, can be expressed in simplest form as $a\sqrt{b}$. Compute a^b.

Time: 10 minutes

5) Point P is a vertex of a cube with edge length 2. Three faces of the cube do not contain point P. The distance from point P to the intersection of the diagonals of one of these three faces of the cube, in simplest form, is $\sqrt{k}$. Compute k.

6) Ten passengers board an empty train of four cars. Each passenger selects, at random, the car in which he will sit. If p is the probability that there will be exactly three passengers in the first car, compute $1000p$ and round to the nearest integer.

Solutions for Contest #4

1) Since 11 is a prime, the only divisors of 11^{12} are powers of 11. There are **13** of them.

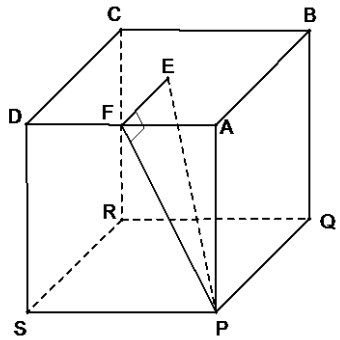
2) Compared with $y = f(x)$, the graph of $y = 3f(x-5)$ is a translation of 5 to the right followed by a tripling of the y-coordinate of that image point. So first, (10,7) becomes (15,7), and then (15,7) becomes (15,21). Thus, $c = 15$, $d = 21$, and $c + d = \mathbf{36.}$

3) In one hour, each tradesperson does one-fourth of the job and each apprentice does one-sixth of the job. So, in one hour, two tradespeople do one-half the job and three apprentices do one-half the job. So, in one hour, the two tradespeople and three apprentices complete the whole job. The number of hours for all of them to do the job together is **1.**

4) The given lines have a slope of 2 and are therefore parallel. The y- intercepts of lines l and m are (0,0) and (0,15), respectively. Through the point (0,15), line $p \perp l$. Therefore, the slope of line p is $-\frac{1}{2}$, its y-intercept is (0,15), and its equation is $y = -\frac{1}{2}x + 15$. Let point Q be the intersection of lines p and l. Solve $-\frac{1}{2}x + 15 = 2x$ to get $x = 6$. Thus, the coordinates of point Q are (6,12). Use the distance formula to get the distance from point Q to (0,15): $\sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$. Then, $a = 3, b = 5$, and the required power is $3^5 = \mathbf{243}$.

   Alternatively, we can use the formula $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$ to find the distance from the point (0, 15) on line m to $2x - y = 0$, which is line l. So, $d = \frac{|2(0) - 1(15) + 0|}{\sqrt{2^2 + (-1)^2}} = \frac{15}{\sqrt{5}} = 3\sqrt{5}$.
   Thus, $3^5 = \mathbf{243}$.

5) In the cube shown in the diagram, let point E be the intersection of the face diagonals $\overline{AC}$ and $\overline{BD}$. Further, let $\overline{EF}$ be perpendicular to $\overline{DA}$, making F the midpoint of $\overline{DA}$. In right $\Delta PAF, PA = 2$ and $AF = 1$, so by the Pythagorean Thm., $PF = \sqrt{5}$. So now, in right ΔPEF, $PF = \sqrt{5}$ and $EF = 1$. Thus, by the Pythagorean Thm. again, $PE = \sqrt{6}$ and $k = \mathbf{6}$.
   Alternatively, if we set up a 3-space coordinate system with the origin at point R, the coordinates of points P, A, and C are (2, 2, 0), (2, 2, 2), and (0, 0, 2), respectively. Since point E is the common midpoint of $\overline{AC}$ and $\overline{BD}$, its coordinates are (1, 1, 2). Thus, by the 3-space distance formula, $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$, $PE = \sqrt{1 + 1 + 4} = \sqrt{6}$. So, $k = \mathbf{6}$.



6) The first passenger can choose any of the four cars. The second passenger can then likewise choose any of the four cars, so that the total number of ways two passengers can choose cars in which to sit is 4^2. Similarly, the number of ways ten passengers can choose cars in which to sit is 4^{10}. There are therefore 4^{10} equally likely possible outcomes. The number of favorable outcomes is determined as follows: The three passengers who sit in the first car can be chosen from the ten passengers in ${}_{10}C_3$ ways. The number of ways the remaining seven passengers can select the other three cars is 3^7. Therefore, the probability that exactly three passengers will be in the first car is
   $p = \frac{{}_{10}C_3 \cdot 3^7}{4^{10}} \approx 0.25028$ and $1000p \approx \mathbf{250}$ to the nearest integer.

Nassau County Interscholastic Mathematics League

Contest #5 Answers must be integers from 0 to 999, inclusive. 2015 – 2016

No calculators are allowed.

Time: 10 minutes

1) Compute the 2015th digit to the right of the decimal point in the decimal representation of $\frac{5}{101}$.

2) A circle whose radius is 4 is tangent to the positive x- axis and the positive y- axis. Two other circles with radii r and R are also tangent to this circle and tangent to the positive x- axis and the positive y- axis. Compute rR.

.

Time: 10 minutes

3) If Isaac gets a 99 on his next mathematics test, then his average will be exactly 92. If he gets a 75, then his average will be exactly 89. Compute the number of mathematics tests Isaac has already taken.

4) Compute the only root of $|x-11|-|x-19|=24-x$.

Time: 10 minutes

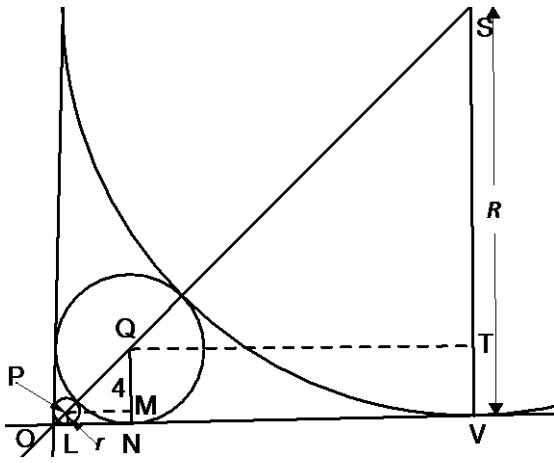
5) The interval on which $(0.5)^{x^2-7x+10} > 1$ is $a < x < b$. Compute ab.

6) Compute the numerical value, in degrees, of $\text{Sin}^{-1}\left(\frac{4}{5}\right) - \text{Tan}^{-1}\left(\frac{1}{7}\right)$, where $\text{Sin}^{-1}(x)$ and $\text{Tan}^{-1}(x)$ are each single-valued expressions that belong to their defined intervals (Note: This means that $\text{Sin}^{-1}(x)$ and $\text{Tan}^{-1}(x)$ are principal-valued expressions).

Solutions for Contest #5

1) The decimal representation of $5/101$ is $0.\overline{0495}$. When 2015 is divided by 4, the remainder is 3, so the 2015^{th} digit to the right of the decimal point is the same as the 3^{rd} digit to the right of the decimal point. Thus, the answer is **9.**

2) In the diagram, the radii of the three circles are (in increasing size order): r, 4, and R. Points O, P, Q, and S (the origin and the centers of the circles in increasing size order) are collinear. From points P, Q, and S, drop perpendiculars to the x- axis. Draw $\overline{PM} \perp \overline{QN}$ and $\overline{QT} \perp \overline{SV}$. $\Delta PQM \sim \Delta QST$, so $\frac{PQ}{QS} = \frac{QM}{ST} \longrightarrow$ $\frac{r+4}{R+4} = \frac{4-r}{R-4} \longrightarrow rR - 4r + 4R - 16 = 4R + 16 - rR - 4r \longrightarrow$ $2rR = 32 \longrightarrow rR = \mathbf{16}$. Alternatively, $OQ = r\sqrt{2} + r + 4 = 4\sqrt{2} \longrightarrow r = \frac{4\sqrt{2}-4}{\sqrt{2}+1}$. Also, $OS = OQ + QS = 4\sqrt{2} + 4 + R = R\sqrt{2} \longrightarrow R = \frac{4\sqrt{2}+4}{\sqrt{2}-1}$. So, $\left(\frac{4\sqrt{2}-4}{\sqrt{2}+1}\right)\left(\frac{4\sqrt{2}+4}{\sqrt{2}-1}\right) = \frac{32-16}{2-1} = \mathbf{16}$.



3) Suppose that Isaac has already taken x mathematics tests and that his average on these tests is y. Then, $\frac{99+xy}{x+1} = 92$ and $\frac{75+xy}{x+1} = 89$. So, $92(x+1) = 99 + xy$ and $89(x+1) = 75 + xy$. Subtracting these last two equations gives $3(x+1) = 24$. So, $x + 1 = 8$ and $x = \mathbf{7}$.

4) Use the definition of absolute value to consider the 3 cases:
If $x \leq 11$, the given equation becomes $-x + 11 - (-x + 19) = 24 - x$, so $x = 32$ (reject)
If $11 < x < 19$, the given equation becomes $x - 11 - (-x + 19) = 24 - x$, so $x = 18$.
If $x \geq 19$, the given equation becomes $x - 11 - (x - 19) = 24 - x$, so $x = 16$ (reject).
Thus, $x = \mathbf{18}$ is the only root. (Note: Although the answer is found using the second case, we show all three cases for completeness.)

5) Let $a = x^2 - 7x + 10$. Then, the given inequality is equivalent to $\left(\frac{1}{2}\right)^a > 1$ or $2^a < 1$. This inequality is true when $a < 0$. So now, solve $x^2 - 7x + 10 < 0$ to get $(x-5)(x-2) < 0$ which yields $2 < x < 5$. Thus, $a = 2$, $b = 5$, and $ab = \mathbf{10}$.

6) Let $A = \text{Sin}^{-1}\left(\frac{4}{5}\right)$ and let $B = \text{Tan}^{-1}\left(\frac{1}{7}\right)$. Use $\sin(A-B) = \sin A \cos B - \cos A \sin B =$ $\frac{4}{5}\cdot\frac{7}{5\sqrt{2}} - \frac{3}{5}\cdot\frac{1}{5\sqrt{2}} = \frac{25}{25\sqrt{2}} = \frac{1}{\sqrt{2}}$. Thus, $A - B = \text{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$ and the required answer is **45**. Note that since $\text{Sin}^{-1}(x)$ and $\text{Tan}^{-1}(x)$ are single-valued and defined only on the intervals $[-90°, 90°]$ and $(-90°, 90°)$, respectively, A, B, and $A - B$ are all first-quadrant angles.
Alternatively, $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{4}{3} - \frac{1}{7}}{1 + \left(\frac{4}{3}\right)\left(\frac{1}{7}\right)} = \frac{25}{25} = 1$. Since $\tan 45° = 1$, then $A - B = \mathbf{45}$.

Nassau County Interscholastic Mathematics League

Team Contest Answers must be integers from 0 to 999, inclusive. 2015 – 2016

Calculators are allowed.

Time: 40 minutes

1) If $x+1$ is a multiple of 4 and $x+3$ is a multiple of 5, compute the remainder when x is divided by 10.

2) For all real numbers x, $15f(x)+f(16-x)=x^3$. Compute $f(8)$.

3) Compute the product of the roots of $|x-1|^2-11|x-1|=-28$.

4) Compute the number of six-digit positive integers that have the following property: The sum of the sixth powers of their digits is less than 64.

5) A square whose sides have length 2 is inscribed in a right triangle so that the right angles coincide. If the length of the hypotenuse of the right triangle is $3\sqrt{5}$, compute the length of the longer leg.

6) Compute the sum of the cubes of the roots of the equation $x^2-11x+23=0$.

7) Compute $\left(\log_3 \sqrt[3]{16}\right)\left(\log_7 \sqrt{27}\right)(\log_5 49)(\log_2 625)$.

8) The numerical value of $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdot\cdots\cdot\left(1-\frac{1}{196}\right)$ is expressed in simplest form as $\frac{p}{q}$. Compute pq.

9) The sum of all numbers of the form $\frac{p}{q}$, where p and q are relatively prime positive integer divisors of 30, can be expressed in simplest form as $\frac{abc}{d}$, where $a, b,$ and c are prime numbers. Compute $a+b+c+d$.

10) In isosceles ΔPQR, $\overline{QN}$ is the altitude to base $\overline{PR}$. Point M is the midpoint of $\overline{QN}$. If $\overleftrightarrow{PM}$ intersects $\overline{QR}$ in point S, compute the quotient of the area of ΔPQR to the area of ΔPQS.

Solutions for Team Contest

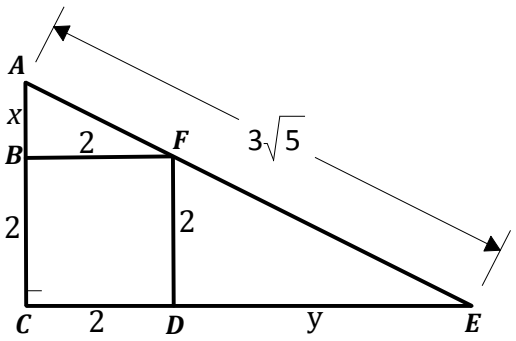
1) Since $x+1$ is even, then $x+3$ is also even. Since $x+3$ is an even multiple of 5, its units digit is 0. Therefore, x has a units digit of 7. Thus, the required remainder is **7.**
Alternatively, the possible values of x when $x+1$ is a multiple of 4 are: 3, 7, 11, 15, 19, 23, 27,… , and the possible values of x when $x+3$ is a multiple of 5 are: 2, 7, 12, 17, 22, 27, … . The numbers that are common to both lists have a units digit of 7. So, when these numbers are divided by 10, their remainder is **7**.

2) Substitute 8 for x in the given equation:
$15f(8)+f(8)=8^3$. Then, $16f(8)=512$ and $f(8)=$ **32**.

3) Let $a=|x-1|$. Then $a^2-11a+28=0$; $(a-7)(a-4)=0$; $a=7$ or $a=4$;
$|x-1|=7$ or 4. Therefore, $x=8,-6,5,$ or -3. The product of these roots is **720**.

4) Since $2^6=64$, the leading digit must be a 1 and each of the other five digits may be 0 or 1. Thus, there are 2^5 or **32** possible numbers with the given property.

5) In the accompanying diagram, square $BFDC$ is inscribed in right triangle ACE with point B on leg $\overline{AC}$, point D on leg $\overline{CE}$, and point F on hypotenuse $\overline{AE}$. Let $AB=x$ and $DE=y$. Triangles ABF and FDE are similar and their corresponding sides are in proportion. Thus, $\frac{x}{2}=\frac{2}{y}$ or $y=\frac{4}{x}$. Since we require the answer to be an integer, the only possible values of x and y are 2 and 2, or 1 and 4 (in either order). If we test each pair, we see that when $x=1$ and $y=4$, the lengths of the legs are 3 and 6, and by the Pythagorean Theorem, $3^2+6^2=\left(3\sqrt{5}\right)^2=45$ is true. Therefore, the length of the longer leg is **6.**



Alternatively, in ΔACE, by the Pythagorean Theorem, $(x+2)^2+\left(2+\frac{4}{x}\right)^2=\left(3\sqrt{5}\right)^2$. Then, $x^2+4x+4+4+\frac{16}{x^2}+\frac{16}{x}=45$ and $x^4+4x^3-37x^2+16x+16=0$. Synthetic division yields $(x-4)(x-1)(x^2+9x+4)=0$. The only nonnegative solutions are 4 and 1. The lengths of the legs are 6 and 3 and so the longer leg has a length of **6**.

6) Note that the sum of the roots of the given equation is 11 and the product of the roots of the given equation is 23. Let the roots of the given equation be r and s. Note that $(r+s)^3=r^3+3r^2s+3rs^2+s^3=r^3+s^3+3rs(r+s)$.
Thus, $r^3+s^3=(r+s)^3-3rs(r+s)=11^3-3\cdot 23\cdot 11=$ **572.**

7) Use the base-change law to change the given expression to $\frac{\log 16^{1/3}}{\log 3}\cdot\frac{\log 27^{1/2}}{\log 7}\cdot\frac{\log 49}{\log 5}\cdot\frac{\log 625}{\log 2}=$
$\frac{\log 2^{4/3}}{\log 2}\cdot\frac{\log 3^{3/2}}{\log 3}\cdot\frac{\log 7^2}{\log 7}\cdot\frac{\log 5^4}{\log 5}=\frac{4}{3}\cdot\frac{3}{2}\cdot 2\cdot 4=$ **16**.

8) Each factor of the given product may be factored as the difference of two squares, and so the given product equals $\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\cdot\cdots\cdot\left(1-\frac{1}{14}\right)\left(1+\frac{1}{14}\right)$
$=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\ldots\cdot\frac{13}{14}\cdot\frac{15}{14}=\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{13}{14}\right)\cdot\left(\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\ldots\cdot\frac{15}{14}\right)=\frac{1}{14}\cdot\frac{15}{2}=\frac{15}{28}$.
Thus, $p=15,\ q=28$ and $pq=\mathbf{420}$.
Alternatively, without factoring the factors of the original product, the sequence of the 13 partial products is $\frac{3}{4},\frac{2}{3},\frac{5}{8},\frac{3}{5},\frac{7}{12},\ldots$. These are equivalent to $\frac{3}{4},\frac{4}{6},\frac{5}{8},\frac{6}{10},\frac{7}{12},\ldots$. So now, the 13th (and last) product is $\frac{15}{28}$. Thus, $15\cdot 28=\mathbf{420}$.

9) The sum of all numbers of the form $\frac{p}{q}$, where p and q are relatively prime positive integer divisors of 30, may be expressed as $\sum_{\substack{a=-1\\b=-1\\c=-1}}^{1}(2^a\cdot 3^b\cdot 5^c)=\frac{1}{2\cdot 3\cdot 5}\sum_{\substack{a=0\\b=0\\c=0}}^{2}(2^a\cdot 3^b\cdot 5^c)=$
$\frac{1}{30}\sum_0^2 2^a\sum_0^2 3^b\sum_0^2 5^c=\frac{1}{30}(1+2+4)(1+3+9)(1+5+25)=\frac{7\cdot 13\cdot 31}{30}$. The required sum is $7+13+31+30=\mathbf{81}$.

10) Without loss of generality, we can let isosceles ΔPQR be an equilateral triangle with a side length of 2. Since an altitude of an equilateral triangle goes to the midpoint of the opposite side, $PN=1$, and because ΔPQN is a 30-60-90 triangle, $MN=QM=\frac{\sqrt{3}}{2}$ and $m\measuredangle PQN=m\measuredangle RQN=30$. Use the Pythagorean Theorem in ΔPMN to get $PM=\frac{\sqrt{7}}{2}$. Let $m\measuredangle PMQ=\theta$ and use the Law of Sines in ΔPQM to yield $\frac{\sin\theta}{2}=\frac{\sin 30}{\frac{\sqrt{7}}{2}}\longrightarrow$
$\sin\theta=\frac{2}{\sqrt{7}}=\sin(180-\theta)$, where $180-\theta=m\measuredangle QMS$. Using the Law of Cosines in ΔPQM, $4=\frac{3}{4}+\frac{7}{4}-2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{7}}{2}\right)\cos\theta\longrightarrow 4=\frac{5}{2}-\frac{\sqrt{21}}{2}\cos\theta\longrightarrow\cos\theta=-\frac{\sqrt{3}}{\sqrt{7}}$. In ΔQSM, $m\measuredangle QSM=\theta-30$, and the Law of Sines in ΔQSM gives $\frac{QS}{\sin(180-\theta)}=\frac{\frac{\sqrt{3}}{2}}{\sin(\theta-30)}$. Thus,
$QS=\frac{\frac{\sqrt{3}}{2}\cdot\frac{2}{\sqrt{7}}}{\sin\theta\cos 30-\cos\theta\sin 30}=\frac{\frac{\sqrt{3}}{\sqrt{7}}}{\frac{2}{\sqrt{7}}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{\sqrt{7}}\cdot\frac{1}{2}}=\frac{\frac{\sqrt{3}}{\sqrt{7}}}{\frac{3\sqrt{3}}{2\sqrt{7}}}=\frac{2}{3}$. Since ΔPQR and ΔPQS have a common altitude from vertex P to the line containing $\overline{QR}$ and $\overline{QS}$, their areas are to each other as the ratio of their bases. So, $\frac{QR}{QS}=\frac{2}{\frac{2}{3}}=\mathbf{3}$.
Alternatively, in the accompanying diagram, points D and B are midpoints of $\overline{QB}$ and $\overline{QR}$, respectively. If we use $|PQR|$ to represent the area of ΔPQR, then $\frac{|PQR|}{|PQB|}=\frac{2}{1}$ and $\frac{|PQR|}{|PQD|}=\frac{4}{1}$. Therefore, since we need an integer solution, $\frac{|PQR|}{|PQS|}=\frac{3}{1}=\mathbf{3}$.

