<u>Solutions for Team Contest</u>

1) As a decimal, $\frac{2}{7} = 0.\overline{285714}$. The repeating part contains six digits. Since $2015 = 6\,(335) + 5$, we are looking for the fifth digit of the repeating part, or **1**.

2) Substitute in the given equation, $x = 1$ and $y = 1 \longrightarrow f(1) = f(1) + 2f(1) \rightarrow f(1) = 0$.
Substitute in the given equation $x = 1$ and $y = 2 \longrightarrow f(2) = f(1) + 2f(2) \rightarrow f(2) = 0$.
Substitute in the given equation $x = 1$ and $y = 3 \longrightarrow f(3) = f(1) + 2f(3) \rightarrow f(3) = 0$.
The pattern continues, so $f(2015) = \mathbf{0}$.

3) The dimensions of the second hole are twice those of the first. So the volume of the second hole is 8 times that of the first. Therefore, two people working at the same rate will take 4 times as long to dig, or **120** minutes.

4) Inspection quickly yields $x = 2$. Otherwise, treat the problem as a quadratic equation in 5^x, where $(5^x)^2 - 5^x - 600 = 0 \longrightarrow (5^x - 25)(5^x + 24) = 0$. Only the first factor has a real solution when set equal to 0, so $5^x = 25 \longrightarrow x = \mathbf{2}$.

5) Note first that when $8 = 2^3$ is divided by 7, the remainder is 1. So, any power of 2^3 also leaves a remainder of 1 when divided by 7. Now, $2^{2013} = (2^3)^{671}$ is expressible as $7k + 1$ for some positive integer k. So, $2^{2014} = 2(7k + 1) = 14k + 2$. So, when 2^{2014} is divided by 7, the remainder is **2.**

<u>Alternate Solution:</u> The pattern of remainders when consecutive powers of 2 are divided by 7 is 2, 4, 1,… repeated forever. Since $2014 = 671 \cdot 3 + 1$, there wll be 671 complete cycles of remainders, and the next remainder will be the start of the next cycle, or 2.

6) $\frac{1}{r^2s} + \frac{1}{rs^2} = \frac{s}{r^2s^2} + \frac{r}{r^2s^2} = \frac{s+r}{r^2s^2}$. Use the facts that the numerator is the sum of the roots of the given quadratic equation, or 1024, and the denominator is the square of the product of the roots, or 4. The quotient, $\frac{1024}{4}$ is the required answer, or **256.**

7) $x^2 - y^2 = 3xy \longrightarrow x^2 - 3xy - y^2 = 0 \longrightarrow \left(\frac{x}{y}\right)^2 - 3\left(\frac{x}{y}\right) - 1 = 0 \longrightarrow \frac{x}{y} = \frac{3 \pm \sqrt{13}}{2}$ by using the Quadratic Formula. The required sum is $3 + 13 + 2 = \mathbf{18}$.

8) Let $QP = x$ and $RP = y$. Therefore, $PS = 20 - x$ and $PT = 16 - y$. Use the formula for the area of a triangle as $\frac{1}{2}$ the product of the lengths of two sides and the sine of the included angle, where $\sin 30° = \sin 150° = \frac{1}{2}$. Using absolute value for area,
$|QRST| = |PRQ| + |PRS| + |PQT| + |PTS| = \frac{1}{2}xy\frac{1}{2} + \frac{1}{2}y(20 - x)\frac{1}{2} + \frac{1}{2}x(16 - y)\frac{1}{2} + \frac{1}{2}(20 - x)(16 - y)\frac{1}{2} = \frac{1}{4}(xy + 20y - xy + 16x - xy + 320 - 16x - 20y + xy) = \mathbf{80.}$

9) Note that $x \neq 0$ and $x \neq -3$. Factor: $\frac{x-1}{\sqrt{x}\sqrt{3+x}} = \frac{\sqrt{x-3}}{\sqrt{x}}$. Reduce: $\frac{x-1}{\sqrt{3+x}} = \frac{\sqrt{x-3}}{1}$. Cross multiply: $\sqrt{x^2 - 9} = x - 1$. Square: $x^2 - 9 = x^2 - 2x + 1$. So, $2x = 10 \longrightarrow x = \mathbf{5}$ and this checks in the original equation.

<u>Alternate Solution</u>: Cross multiply and square both sides of the equation, yielding $x(x-1)^2 = x(x-3)(x+3)$ with solutions $x = 5$ or $x = 0$. Reject $x = 0$ because it makes both denominators 0.

10) $P(\text{first 5 occurs on toss \#1}) + P(\text{first 5 occurs on toss \#3}) + P(\text{first 5 occurs on toss \#5}) + \cdots$
$= \frac{1}{6} + \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right) + \cdots$. This is a geometric series whose first term is $a = \frac{1}{6}$ and whose ratio is $r = \frac{25}{36}$. Using the formula $S = \frac{a}{1-r}$, the sum of the series, $S = \frac{1/6}{1-25/36} = \frac{6}{11}$.
Thus, the answer is $6 + 11 = \mathbf{17}$.