Time: 10 minutes

1) Two 8 x 8 squares overlap to form an 8 x 10 rectangle. Find the area of the region common to the interiors of both squares.

2) Find the smallest natural number x that satisfies: $\sqrt{x^2 + 20x + 99} > 30$.

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Time: 10 minutes

3) If $P(x)$ is a polynomial such that $x = 11$ is a solution of $P(x) - x = 0$, compute $P\big(P(P(11))\big)$.

4) Find the smallest positive integer which leaves a remainder of 3 when divided by 4, a remainder of 3 when divided by 5, and a remainder of 8 when divided by 11.

Time: 10 minutes

5) If Sophia gets 100 on her next test, her average will be 91. If she gets 76, her average will be 88. How many tests has Sophia already taken?

6) A set of townhouses has three mailboxes. The mail deliverer is trying to finish her route quickly and without concern for necessarily placing the mail in the proper boxes. If her only concern is placing at least one piece of mail in each box, compute the number of ways she can do this with exactly 6 pieces of mail.

<u>Solutions for Contest #5</u>
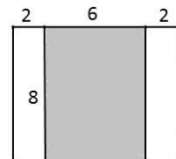
1) The area of the shaded region common to both squares is $6 \cdot 8 = \mathbf{48}$.

2) Complete the square: $\sqrt{x^2 + 20x + 100 - 1} > 30$. Re-write: $\sqrt{(x+10)^2 - 1} > 30$. Therefore, $(x+10)^2 - 1 > 900 \rightarrow (x+10)^2 > 901$. Since x is a natural number, $x + 10 > 30 \rightarrow x > 20$. The required answer is $\mathbf{21}$.

3) Since $P(x) = x$, $P(11) = 11$. Successive iterations yield $\mathbf{11}$.

4) Suppose the number we seek is x. Then $x - 3$ must be a multiple of 4 and 5 and therefore, 20. So x must be of the form $20n + 3$, where n is a positive integer. It also follows that x must be of the form $11m + 8$, where m is a positive integer. Thus, $20n + 3 = 11m + 8 \rightarrow 20n = 11m + 5$. The smallest possible value of m is 5, yielding $n = 3$. This gives $x = \mathbf{63}$.

   Note: The original wording of the problem made it subject to appeal: Find the smallest positive integer which leaves a remainder of 3 when divided by 4 **or** 5, and a remainder of 8 when divided by 11.

   With this wording, the accepted answer was 8.

5) Let Sophia's average for her first $n - 1$ tests be a. Then, $(n-1)a + 100 = 91n$ and $(n-1)a + 76 = 88n$. Subtract the second equation from the first to get $24 = 3n$ and $n = 8$. So, Sophia had already taken $\mathbf{7}$ tests. Note: An alternate approach would be to let n be the number of tests Sophia already took.

6) Case I: She places four pieces of mail in one box and one piece in each of the remaining two boxes. The four pieces of mail can go in any of three boxes. The four pieces of mail can be chosen in $\binom{6}{4} = 15$ ways. Then, there are two ways to distribute the remaining two pieces of mail. This is a total of $3 \cdot 15 \cdot 2 = 90$ ways.

   Case II: She places three pieces of mail in one box, two in another, and one in the remaining box. This can be arranged in 6 ways. She can choose three pieces of mail in $\binom{6}{3} = 20$ ways. Then she can choose two of the remaining pieces of mail in $\binom{3}{2} = 3$ ways, and then the last letter is placed in the remaining box for a total of $6 \cdot 20 \cdot 3 = 360$ ways.

Case III: She places two letters in each of three boxes. First she chooses two pieces of mail out of six, then two out of four, and then places the last two letters in the remaining box for a total of $\binom{6}{2}\binom{4}{2}\binom{2}{2} = 90$ ways.

Thus, $90 + 360 + 90 =$ **540**.

Alternate Solution:

The number of ways of distributing 6 letters into 3 mailboxes is $3^6 = 729$ because for each letter we have 3 choices.

We need to subtract from this all the ways that leave one or more mailboxes empty.

If exactly one mailbox is empty, we can place the letters in the other mailboxes in 2^6 - 2 = 62 ways. (subtract 2 because we do not want to leave two boxes empty)

If two mailboxes are empty, then all the letters are placed in one box (1 way to do this).

$729 - 3(62) - 3 = 540$