Nassau County Interscholastic Mathematics League

Contest #3 Answers must be integers from 0 to 999, inclusive. 2014 – 2015

No calculators are allowed.

Time: 10 minutes

1) What two-digit number is exactly twice the sum of its digits?

2) Sides $\overline{AB}$ and $\overline{CD}$ of regular hexagon $ABCDEF$ are extended to meet in point G. If $AB = 2$ and $EG = a\sqrt{b}$, where $\sqrt{b}$ is in simplest form, compute $a + b$.

Time: 10 minutes

3) If Isaac walks to school and returns by bus, the entire trip takes 90 minutes. If he takes the bus both ways it takes him 30 minutes. Isaac walks at a constant rate and the bus rides at a constant rate. How many minutes will it take Isaac to walk both ways?

4) In a circle, parallel chords with lengths of 8 and 12 inches are 10 inches apart. The center of the circle is between the chords. If the radius of the circle is $\sqrt{k}$, compute k.

Time: 10 minutes

5) For how many pairs (x, y) of positive integers does $x^2 - y^2 = 315$?

6) If $x > 0$ and $y > 0$ and $x^2 + y^2 = 98xy$, then $\log(x + y)$ can be expressed as $A \log x + B \log y + C$, where A, B, and C are real numbers, and all logarithms are base-ten logarithms. Compute $100ABC$.

Solutions for Contest #3

1) If t is the tens digit and u is the ones digit, then $10t + u = 2(t + u) \rightarrow u = 8t$. The maximum possible value for t is 1. Therefore, $u = 8$ and the required number is **18.**

2) Extend $\overline{ED}$ through point D and drop a perpendicular from point G to meet $\overrightarrow{ED}$ at point H. Now, $m\measuredangle CBG = m\measuredangle BCG = m\measuredangle BGC = m\measuredangle HDG = 60°$. So, ΔBGC is equilateral and $GC = 2$, making $GD = 4$, and in 30°-60°-90° $\Delta GDH, GH = 2\sqrt{3}$. Now, in right ΔGEH, $(EG)^2 = (GH)^2 + (HE)^2 = \left(2\sqrt{3}\right)^2 + 4^2 = 28$. So, $EG = 2\sqrt{7}$ and the required sum is **9**.
Alternate solution: Draw $\overline{EG}$, and use the Law of Cosines in ΔEDG:
$(EG)^2 = (ED)^2 + (DG)^2 - 2(ED)(DG)\cos 120° = 2^2 + 4^2 - 2 \cdot 2 \cdot 4 \cdot (-1/2) = 28 \longrightarrow$
$EG = 2\sqrt{7}$, so the required sum is 9.

3) Let
d be the distance from home to school, r_w be Isaac's walking rate, and r_b be the bus' rate. In minutes, $\frac{d}{r_w} + \frac{d}{r_b} = 90$ and $\frac{2d}{r_b} = 30 \longrightarrow \frac{d}{r_b} = 15 \longrightarrow \frac{d}{r_w} = 75 \longrightarrow \frac{2d}{r_w} = \mathbf{150}$ minutes.

Alternate solution: If a two-way bus trip takes 30 minutes, a one-way trip must take 15 minutes. Thus Isaac takes $90 - 15 = 75$ minutes to walk to school one way, so a two-way trip would take 150 minutes.

4) Without loss of generality, name the shorter chord $\overline{AB}$ and the longer chord $\overline{CD}$ in circle O. Draw radii $\overline{OA}$ and $\overline{OC}$, each with length r. Drop perpendiculars from point O to point M in chord $\overline{CD}$ and to point N in chord $\overline{AB}$. Let $OM = x$ and $ON = 10 - x$. In the right triangles, we have $16 + (10 - x)^2 = r^2$ and $36 + x^2 = r^2$. Solving simultaneously yields $x = 4$ and $r^2 = 52$. The required answer is **52**.

5) When factored in pairs, $315 = 1 \cdot 315 = 3 \cdot 105 = 5 \cdot 63 = 7 \cdot 45 = 9 \cdot 35 = 15 \cdot 21$ If, for example, $x - y = 1$ and $x + y = 315$, then $(x, y) = (158,157)$. Proceeding in a similar manner with the other products,
$(x, y) = (54, 51), (34, 29), (26,19), (22,13)$, or $(18,3)$. There are **6** such ordered pairs.

6) Complete the square on the left side of the original equation and re-write as the square of a binomial: $(x + y)^2 = 100xy \longrightarrow 2\log(x + y) = 2 + \log x + \log y \longrightarrow$
$\log(x + y) = 1 + \frac{1}{2}\log x + \frac{1}{2}\log y$. So, $100ABC = 100 \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \mathbf{25}$