Nassau County Interscholastic Mathematics League

Contest #2 Answers must be integers from 0 to 999, inclusive. 2014 – 2015

Calculators are allowed.

Time: 10 minutes

1) The average age of the eleven members of a team is 27. During a game, one player is ejected and not replaced. The average age of the remaining players is 26. What is the age of the ejected player?

2) Points D, E, and F are chosen respectively on sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ of ΔABC. If $AD:DB = BE:EC = CF:FA = 1:3$ and the ratio of the area of ΔDEF to the area of ΔABC is expressed as p/q in simplest form, compute $p+q$.

Time: 10 minutes

3) How many two-digit numbers are there such that the sum of the digits is a perfect square or a perfect cube?

4) Point X is in $\overline{AB}$ with $AX:XB = 3:4$. Point Y is in $\overline{AB}$ with $AY:YB = 4:5$. If $XY = 2$, find AB.

Time: 10 minutes

5) Three adjacent faces of a rectangular parallelepiped (box) have areas of 5, 10, and 18. Compute the volume of the solid.

6) Circle O is tangent to $\overline{RS}$ of equilateral ΔRST at its midpoint and has no point interior to ΔRST. If circle O is also tangent to circle P which circumscribes ΔRST, $RS = 2$ and the area of circle O is $\frac{\pi}{k}$, compute k.

<u>Solutions for Contest #2</u>

1) The sum of the players' ages was $27 \cdot 11 = 297$. The sum of the ages of the remaining players is $26 \cdot 10 = 260$. So, the ejected player is $297 - 260 =$ **37** years old.

2) Use the absolute value symbol for area as well as the formula: $K = \frac{1}{2}ab\sin C$.
$\frac{|BED|}{|ABC|} = \frac{|FCE|}{|ABC|} = \frac{|FAD|}{|ABC|} = \frac{\frac{1}{2}FA \cdot AD \sin A}{\frac{1}{2}CA \cdot AB \sin A} = \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$. Therefore, $\frac{|FAD|+|FCE|+|BED|}{|ABC|} = \frac{9}{16}$.
So, $\frac{|DEF|}{|ABC|} = 1 - \frac{9}{16} = \frac{7}{16}$. The required sum is **23.**

3) The largest possible digital sum is 18. Perfect squares less than or equal to 18 are 1, 4, 9, and 16. Perfect cubes are 1 and 8. The only two-digit numbers meeting the required conditions are 10, 40, 80, 90, 13, 31, 22, 17, 71, 26, 62, 35, 53, 44, 18, 81, 27, 72, 36, 63, 45, 54, 79, 97, and 88. There are **25** such numbers.

4) Point X is 3/7 of the way from point A to point B, and point Y is 4/9 of the way from A to B. Therefore, the points are in the following order: $\overline{AXYB}$. Let $AX = 3a$, $XB = 4a$, $AY = 4b$ and $YB = 5b$. So, $7a = 9b$ and $b = \frac{7a}{9}$. Also, $XY + YB = XB$, or $2+5b = 4a$.
Substitution yields $2 + \frac{35a}{9} = 4a \longrightarrow a = 18$, and $AB = 7 \cdot 18 = \mathbf{126.}$
<u>Alternate solution:</u> Let $c = AX$ and $d = AY$, so that $d - c = XY$. Then,
$d - c = (4/9)AB - (3/7)AB = (1/63)AB = 2 \longrightarrow AB = 126$.

5) Let the lengths of the edges of the solid be x, y, and z. Then, $xy = 5$, $yz = 10$, and $xz = 18$. Multiplication yields $(xyz)^2 = 900$ and $xyz = 30$. So, the volume is **30.**

6) Let point M be the midpoint of $\overline{RS}$. Let r be the radius of circle O. Let point N be the point where the two circles are tangent to each other forming diameter $\overline{NOMPT}$ of circle P. On that diameter, $2r + PM = PT = \frac{2}{3}TM$ because the medians of a triangle intersect in a 2:1 ratio. Continue with $2r + \frac{1}{3}TM = \frac{2}{3}TM \longrightarrow r = \frac{1}{6}TM$ and $r = \frac{1}{6}\sqrt{3}$ using properties of the 30°-60°-90° ΔTMR. So, the area of circle O is $\pi r^2 = \pi/12$. Thus, the required answer is **12.**