Nassau County Interscholastic Mathematics League

Contest #1 Answers must be integers from 0 to 999, inclusive. 2014 – 2015

No calculators are allowed.

Time: 10 minutes

1) On eight tests whose grades range from 0 to 100, inclusive, Amanda's average is exactly 90. Find the lowest possible score she could have earned on any one of the tests.

2) Find the value of x for which $x! = 2^{22} \cdot 3^{10} \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$, where $x!$ is x factorial.

Time: 10 minutes

3) The ratio of the measure of each interior angle of a regular polygon to the measure of each exterior angle is 17:1. Compute the number of sides of the regular polygon.

4) Quadrilateral $PQRS$ is a square with point T on side $\overline{QR}$ and point V on side $\overline{RS}$, with $PQ = 12$, and $m\measuredangle PTQ = m\measuredangle PVS = 60°$. Compute the area of ΔPTV.

Time: 10 minutes

5) If $f(x) = \frac{1}{1-x}$ with $x > 1$ and $g(x) = 1 - \frac{1}{x}$, find $g\big(f(5)\big) - f\big(g(5)\big)$.

6) For how many positive integral values of $k < 500$ is $\sqrt{8k+1}$ an integer?

Solutions for Contest #1

1) To have an average of 90 on 8 tests requires a total score of 720. If she scored 100 on each of seven tests, then she could score as low as **20** on the remaining test.

2) The fact that the required product contains consecutive positive integers is key. We consider the exponentiated factors and realize, for example, that one of the 11's will be used for 11, and the other with a factor of 2 will be used for 22. The "missing" factors are accounted for using $4 = 2^2$, $6 = 2 \cdot 3$, $8 = 2^3$, $9 = 3^2$, $10 = 2 \cdot 5$, $12 = 3 \cdot 2^2$, $14 = 2 \cdot 7$, $15 = 3 \cdot 5$,
$16 = 2^4$, $18 = 2 \cdot 3^2$, $20 = 2^2 \cdot 5$, $21 = 3 \cdot 7$, $22 = 2 \cdot 11$, $24 = 2^3 \cdot 3$ So, $x = \mathbf{24}$.

3) An interior angle of a convex polygon and its adjacent exterior angle are supplementary, so, $17x + x = 180 \longrightarrow x = 10$ and $\frac{360}{10} = \mathbf{36}$.

4) Triangles PTQ and PVS are 30°-60°-90° triangles. So, $QT = SV = \frac{12}{\sqrt{3}}$, $PT = PV = \frac{24}{\sqrt{3}}$, and $m\measuredangle TPV = 30°$. The area of $\Delta PTV = \frac{1}{2} \cdot \frac{24}{\sqrt{3}} \cdot \frac{24}{\sqrt{3}} \sin 30° = \mathbf{48}$.

5) $g\big(f(5)\big) - f\big(g(5)\big) = g\left(-\frac{1}{4}\right) - f\left(\frac{4}{5}\right) = 5 - 5 = \mathbf{0}$. Also note that the two functions are inverses of each other. Hence, $f\big(g(x)\big) = g(f(x)) = x$, yielding the same result.

6) Make a table by setting $8k + 1$ equal to a perfect square and notice a pattern:

$8k+1$	1	4	9	16	25	36	49	64	81	100	121	144
$(8k+1)\text{mod}8$	1	4	1	0	1	4	1	0	1	4	1	0
k	0		1		3		6		10		15	

Notice from the table that k is a sequence of triangular numbers when $8k + 1$ is the square of an odd number. We can justify this as follows: For positive integer n, $8k + 1 = (2n + 1)^2 \longrightarrow 8k = 4n^2 + 4n \longrightarrow k = \frac{4n(n+1)}{8} \longrightarrow k = \frac{n(n+1)}{2}$. We are required to have $\frac{n(n+1)}{2} < 500$, or $n(n + 1) < 1000$. The largest integral n that satisfies this condition is **31**.

<u>Alternate solution:</u> Set $8k + 1 = n^2$, so $k = \frac{n^2-1}{8}$. If n is odd, let $n = 2m + 1$. Then, $n^2 - 1 = 4m(m + 1)$. Since exactly one of m or $m + 1$ must be even, $n^2 - 1$ is divisible by 8 if (and only if) n is odd. The requirement that $\frac{n^2-1}{8} < 500$ holds for $n < 64$. There are 32 positive odd integers less than 64, but choosing $n = 1$ results in $k = 0$, so there are 31 possible values of k.