

Nassau County Interscholastic Mathematics League
Team Contest Answers must be integers from 0 to 999 inclusive. 2012 – 2013
Calculators are allowed.

1. Find the smallest positive integer that is not a divisor of $31!$
2. Find the sum of the integer solutions of $1 < (x - 5)^2 < 100$.
3. Compute the minimum possible sum of digits for a positive integer that is a multiple of 17.
4. One root of $x^2 + kx - 4 = 0$ is the square of the other root. Find the sum of the cubes of the roots.
5. A tetromino is a figure that consists of four unit squares, each of which shares at least one side with at least one of the other three squares. Compute the number of non-congruent tetrominos.
6. If $\sin x - \cos x = \frac{1}{2}$, and $\sin 2x = \frac{m}{n}$, where m and n are relatively prime positive integers, compute $m + n$.
7. A 2 by 3 rectangle is to be covered by 1 by 2 rectangles and 1 by 1 squares. The 1 by 2 rectangles and 1 by 1 squares must not overlap, and their sides must be parallel to the sides of the 2 by 3 rectangle. Compute the number of possible patterns for such coverings.
8. If $\frac{1}{\sqrt[3]{2}-1} = a + \sqrt[3]{b} + \sqrt[3]{c}$, and a , b , and c are positive integers, **with $a < b < c$** , find $100a + 10b + c$.
9. The centers of circles O and P are inside equilateral triangle ABC , and their radii are 1 and 2, respectively. Circles O and P are externally tangent, circle O is tangent to \overline{AB} and \overline{AC} , and circle P is tangent to \overline{AB} and \overline{BC} . Given that $AB = \sqrt{m} + \sqrt{n}$, where m and n are positive integers, find $m + n$.
10. Find the value of x for which:
$$\left(\frac{1+\sqrt{5}}{2}\right)^{2012} + \left(\frac{1+\sqrt{5}}{2}\right)^{2013} = \left(\frac{1+\sqrt{5}}{2}\right)^x.$$

Solutions for Team Contest

1. The positive integers from 1 to 31 inclusive all divide $31!$. So do 32, 33, 34, 35, and 36 because their prime factors are less than 31. But 37 does not divide $31!$ because 37 is not a prime factor of any of the integers from 1 to 31.

2. Use a transformation. Each of the solutions of $1 < (x - 5)^2 < 100$ is 5 greater than a corresponding solution of $1 < x^2 < 100$. There are 16 solutions of the latter inequality, and their sum is 0. So, the sum of the solutions of the original inequality is $16 \cdot 5 = 80$.

Alternate solution: Since $\sqrt{a^2} = |a|$ for all values of a , the original inequality is equivalent to $1 < |x - 5| < 10$. Thus, the distance on the number line between the point whose coordinate is 5 and the point whose coordinate is represented by x is greater than one and less than ten. The integers that meet these requirements satisfy $(x < 4 \text{ or } x > 6)$ and $-5 < x < 15$. The sum of these sixteen integers is 80.

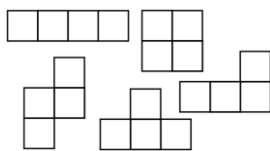
3. A number whose digit-sum is 1 must consist of a 1 followed by any number of 0's (including none), but no number ending in 0 whose digit-sum is 1 can be a multiple of 17. If a number whose digit-sum is 2 were to be a multiple of 17, it would have to consist of two 1's with some number of 0's between them. Check on a calculator to find that 100000001 is a multiple of 17. Thus, the minimum digit-sum is 2.

Alternate solution: Note that $10^2 = 100 \equiv 15 \pmod{17} \equiv -2 \pmod{17}$.

So, $10^8 = (10^2)^4 \equiv (-2)^4 \pmod{17} \equiv 16 \pmod{17}$. Therefore,

$10^8 + 1$ is a multiple of 17.

4. Denote the roots by r and r^2 . Then $-4 = r \cdot r^2$, so $r = \sqrt[3]{-4}$, and so $r^2 = \sqrt[3]{16}$. Thus, the requested sum is $-4 + 16 = 12$.



5. There are five.

6. Square both sides to obtain $\sin^2 x + \cos^2 x - 2\sin x \cos x = \frac{1}{4}$. This is equivalent to $1 - \sin 2x = \frac{1}{4}$, so $\sin 2x = \frac{3}{4}$ and $m + n = 7$.

7. The number of 1 by 2 rectangles can be 0, 1, 2, or 3. Once all of them are placed, there is only one way to place the 1 by 1 squares. When there are 0 such rectangles, there is 1 possible pattern. There are 7 possible placements for 1 such rectangle because there are 2 placements in each of the two rows and 1 in each of the 3 columns. To place 2 rectangles, consider 3 cases: both horizontal (HH),

both vertical (VV), and one horizontal and one vertical (HV). There are 4 placements of the first type, 3 of the second, and 4 of the third for a total of 11. For 3 rectangles, the cases are HHH, HHV, HVV, and VVV, and there are 0, 2, 0, and 1 placements, respectively, for these cases for a total of 3. Thus there are $1 + 7 + 11 + 3 = 22$ possible patterns.

8. Let $x = \sqrt[3]{2}$.

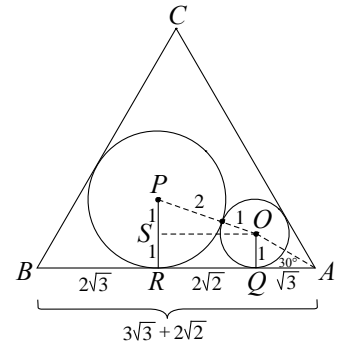
Then $\frac{1}{\sqrt[3]{2}-1} = \frac{1}{x-1} = \frac{x^2+x+1}{(x-1)(x^2+x+1)} = \frac{x^2+x+1}{x^3-1} = x^2 + x + 1 = \sqrt[3]{4} + \sqrt[3]{2} + 1$.

Since $a < b < c$, $a = 1$, $b = 2$, and $c = 4$. Thus, $100a + 10b + c = 124$.

9. Let Q and R be the projections of O and P , respectively, onto \overline{AB} . Notice that $\triangle OAQ$ is a $30^\circ - 60^\circ - 90^\circ$ triangle, so $AQ = \sqrt{3}$. Similarly, $BR = 2\sqrt{3}$. In quadrilateral $OPRQ$, $OP = 1 + 2 = 3$, $PR = 2$, and $OQ = 1$. Let S be the projection of point O onto \overline{PR} . Then $PS = 2 - 1 = 1$.

So, $OS = \sqrt{3^2 - 1} = 2\sqrt{2}$.

Thus, $AB = 3\sqrt{3} + 2\sqrt{2} = \sqrt{27} + \sqrt{8}$. So, $m + n = 35$.



10. Let $p = \frac{1+\sqrt{5}}{2}$. Then, $p^{2012}(1+p) = p^x$ or $1+p = p^{x-2012}$.

So $\frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^{x-2012}$. Therefore, $\frac{3+\sqrt{5}}{2}$ is a power of $\frac{1+\sqrt{5}}{2}$.

So, $x - 2012 = 2$ and $x = 2014$.

Alternate solution: Let $p = \frac{1+\sqrt{5}}{2}$, and let $q = \frac{1-\sqrt{5}}{2}$.

Then $p + q = 1$ and $pq = -1$. So p and q are roots of $x^2 - x - 1 = 0$.

Thus, p must satisfy $p^2 = p + 1$. Multiply both sides by p^{2012} to find

that $p^{2014} = p^{2013} + p^{2012}$, and that therefore $x = 2014$ is a solution of the given equation. The given equation must have a unique solution because $f(x) = p^x$ is an increasing function whose range is the set of positive real numbers.