Nassau County Interscholastic Mathematics League

Contest #5 Answers must be integers from 0 to 999 inclusive. 2013 – 2014

No calculators are allowed.

Time: 10 minutes

1. A proper divisor of an integer is any positive divisor of the integer except for the integer itself. What is the sum of the proper divisors of 140?

2. Quadrilateral $WXYZ$ is a square whose side has a length of three. If square $ABCD$ is inscribed in triangle XYZ with $\overline{AD}$ on $\overline{XZ}$, B on $\overline{XY}$, and C on $\overline{ZY}$, find the area of square $ABCD$.

Time: 10 minutes

3. How many times does the digit 2 appear in the page numbers of a 700-page book?

4. The length of a side of a cube is 10 inches. The area of a cross section of the cube made by a plane through parallel edges NOT in the same face is represented in simplest radical form as $p\sqrt{q}$. Compute $p+q$.

Time: 10 minutes

5. The function $f(x)$ is odd if for all x, $f(-x) = -f(x)$. The function $f(x)$ is even if for all x, $f(-x) = f(x)$. If the functions h and g are defined for all real numbers, $g(x)$ is even and the function $h(x)$ is odd, compute

$$\frac{h(-4)+4g(-4)+4g(4)+h(4)}{h(0)+g(4)}.$$

6. The value of the infinite product: $2^{\frac{1}{5}} \cdot 4^{\frac{1}{25}} \cdot 8^{\frac{1}{125}} \cdot 16^{\frac{1}{625}} \cdot \cdots$ can be expressed in simplest radical form as $\sqrt[n]{a^b}$. Compute $n+a+b$.

Solutions for Contest #5

1. **196**. $140 = 2^2 \cdot 5 \cdot 7$. The proper divisors of 140 are 1, 2, 4, 5, 7, 10, 14, 20, 28, 35, and 70. Their sum is 196.

Method 2: The sum of the divisors of *n*, denoted by $\sigma(n)$, is given by the formula $\prod_{i=1}^{k}(1 + p_i + p_i^2 + \cdots + p_i^{a_i})$ where n is factored into primes as $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$.

Using the above, the sum of the divisors of 140 is $(1 + 2 + 2^2)(1 + 5)(1 + 7) = 336$. The desired sum is 336 – 140 = 196.

2. **2.** Let $AB = x$. Then, $AB = XA = DZ = x$ and $ZX = 3x = 3\sqrt{2}$ So, the area of square $ABCD$=$\left(\sqrt{2}\right)^2 = 2$.

3. **240**. The digit 2 appears as a hundreds digit 100 times, as a tens digit $10 \cdot 7 = 70$ times, and as a units digit $10 \cdot 7 = 70$ times for a total of 240 times.

4. **102.** The required cross section is a rectangle that stretches from the top edge on one side of the cube to the bottom edge on the opposite side of the cube. One pair of opposite sides of the rectangle are diagonals of a face of the cube and the other pair of opposite sides of the rectangle are edges of the cube. The required area is $10 \cdot 10\sqrt{2} = 100\sqrt{2}$. The required sum is 102.

5. **8.** Note that $h(-4) = -h(4), g(-4) = g(4)$, and $h(0) = 0$. So, $\frac{h(-4)+4g(-4)+4g(4)+h(4)}{h(0)+g(4)} = \frac{0+8g(4)}{0+g(4)} = 8$.

6. **23**. The given infinite product $2^{\frac{1}{5}} \cdot 4^{\frac{1}{25}} \cdot 8^{\frac{1}{125}} \cdot 16^{\frac{1}{625}} \cdot \cdots = 2^{\frac{1}{5}} \cdot 2^{\frac{2}{25}} \cdot 2^{\frac{3}{125}} \cdot 2^{\frac{4}{625}} \cdot \cdots = 2^{\left(\frac{1}{5}+\frac{2}{25}+\frac{3}{125}+\frac{4}{625}+\cdots\right)}$.

Let $S = \frac{1}{5} + \frac{2}{25} + \frac{3}{125} + \frac{4}{625} + \cdots$. Then $5S = 1 + \frac{2}{5} + \frac{3}{25} + \cdots$. Subtract these equations to get:

$4S = 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \cdots$ and $4S = \frac{1}{1-1/5} = \frac{5}{4}$ and $S = \frac{5}{16}$. So the given infinite product is equal to $2^{5/16}$ and the required sum is 2 + 5 + 16 = 23.