Nassau County Interscholastic Mathematics League

Contest #3 Answers must be integers from 0 to 999 inclusive. 2013 – 2014

No calculators are allowed.

Time: 10 minutes

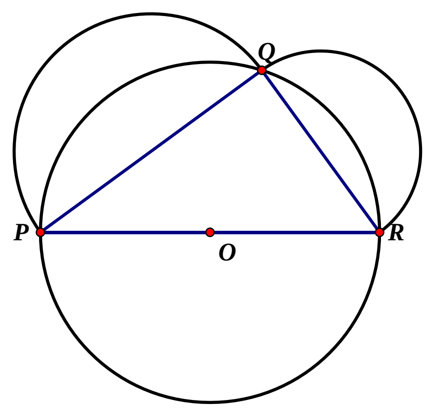
1. The positive integers *p* and *p* + 11 are both prime. Compute *p*.

2. Find k so that the point $(-2, k)$ is on the perpendicular bisector of the line segment whose endpoints are $(-3, -3)$ and $(5,1)$.

Time: 10 minutes

3. Compute the largest integral value of x such that $\frac{32!}{2^x}$ is an integer.

4. Triangle PQR is inscribed in circle O and side $\overline{PR}$ is a diameter of circle O. Semicircles are drawn with diameters $\overline{PQ}$ and $\overline{QR}$ as shown. If $PQ = 8$ and $QR = 6$, compute the sum of the areas interior to the semicircles but exterior to circle O.



Time: 10 minutes

5. Points P and Q are on circle O such that $mPQ = 100°$. A second circle is drawn through points O, P, and Q. Find the degree measure of POQ.

6. In triangle ABC, $AB = 13$, $AC = 14$, and $BC = 15$. If $\sin 2A = \frac{p}{q}$, where $\frac{p}{q}$ is expressed in simplest form, compute *p* + *q*.

Solutions for Contest #3

1. **2.** One of the integers must be even and the only even prime number is 2, so p = 2.

2. **5.** The slope of the given line segment is ½ and its midpoint is $(1, -1)$. The slope of the line joining $(-2, k)$ and $(1, -1)$ is -2. So, $\frac{k+1}{-2-1} = -2$ and $k = 5$.

3. **31.** Thirty-two factorial contains one factor of two for each of 2, 6, 10, 14, 18, 22, 26, and 30.

Thirty-two factorial contains two factors of two for each of 4, 12, 20, and 28.

Thirty-two factorial contains three factors of two for each of 8 and 24.

Thirty-two factorial contains four factors of two for 16 and five factors of two for 32.

Therefore, $x = 1(8) + 2(4) + 3(2) + 4(1) + 5(1) = 31$.

Method 2: Compute $\frac{32}{2} + \frac{32}{4} + \frac{32}{8} + \frac{32}{16} + \frac{32}{32} = 16 + 8 + 4 + 2 + 1 = 31$.

4. **24.** Since $\measuredangle PQR$ is inscribed in a semicircle, it is a right angle and so the Pythagorean Theorem implies that $PR = 10$. To get the required area, we will subtract the quantity (area of triangle PQR from the area of the semicircle with center O) from the sum of the areas of the semicircles with diameters $\overline{PQ}$ and $\overline{QR}$. The required area $= \frac{1}{2}\pi \cdot 3^2 + \frac{1}{2}\pi \cdot 4^2 - \left(\frac{1}{2}\pi \cdot 5^2 - \frac{1}{2} \cdot 6 \cdot 8\right) = 24$

5. **160.** Note that $\measuredangle POQ$ is a central angle in the first circle and an inscribed angle in the second. In the second circle, the measure of inscribed $\measuredangle POQ = 100°$. The measure of its intercepted arc in the second circle is $200°$.
The $mPOQ = 360° - 200° = 160°$.

6. **289.** Use the law of cosines: $15^2 = 13^2 + 14^2 - 2 \cdot 13 \cdot 14 \cos A$. Then $\cos A = \frac{5}{13}$. Hence, $\sin A = \frac{12}{13}$. Then $\sin 2A = 2 \sin A \cos A = 2 \cdot \frac{12}{13} \cdot \frac{5}{13} = \frac{120}{169}$. The required sum is 289.

Method 2: The given triangle is a *heronian triangle* – it has integer sides and integer area. We may partition this triangle into two right triangles (with sides 5,12,13 and 9,12,15) to compute the area to be 84. Then we have:

$\frac{1}{2}bc\sin A = 84 \Rightarrow \frac{1}{2}(13)(14)\sin A = 84 \Rightarrow \sin A = \frac{12}{13}$ and $\sin(2A) = 2\sin A\cos A = 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right) = \frac{120}{169}$