Nassau County Interscholastic Mathematics League

Contest #2 Answers must be integers from 0 to 999 inclusive. 2013 – 2014

Calculators are allowed.

Time: 10 minutes

1. For a given purchase, a discount of 15% is followed by a discount of 10% and the resulting sales price is $45.90. What was the original price, in dollars, before any discount was applied?

2. The coordinates of the vertices of quadrilateral $ABCD$ are $A(-3,6)$, $B(5,8), C(1,-6)$, and $D(-7,-4)$. If the consecutive midpoints of the sides of quadrilateral $ABCD$ are joined to form a new quadrilateral, find the area of the new quadrilateral.

Time: 10 minutes

3. A sphere has a radius of 4 inches. What is the volume, in cubic inches, of the smallest cube that can contain the entire sphere?

4. The distance between the lines $y = \frac{1}{2}x - 5$ and $y = \frac{1}{2}x + 10$ may be expressed in simplest radical form as $p\sqrt{q}$. Compute $p + q$.

Time: 10 minutes

5. If $f(x) = x^3 - 121$ and $f\big(g(x)\big) = \sqrt{x-4}$, compute $g(20)$.

6. Compute $\sum_{k=1}^{50} (-1)^k \cos(k\pi)$.

Solutions for Contest #2

1. **60**. If the original price is *x* dollars, then 0.85 (0.9)x = 45.9 and $x = 60$.

2. **48**. The coordinates of the midpoints are $(1{,}7), (3{,}1), (-3, -5)$, and $(-5{,}1)$. When consecutive midpoints of any quadrilateral are joined, the resulting quadrilateral is always a parallelogram. Notice that the diagonal of this parallelogram joining (-5,1) and (3,1) is a horizontal line. The area of the two congruent triangles above and below this diagonal can be calculated easily: 2[(1/2)(8)(6)] = 48 .

Method 2: Surround the new quadrilateral with a rectangle and calculate its area $(8 \cdot 12 = 96)$. Subtract the sum of the areas of the exterior triangles $(18 + 6 + 18 + 6 = 48)$ from this and the result is 48.

3. **512.** The length of the side of the cube and the length of the diameter of the sphere are both 8 inches. The volume of the cube is $8^3 = 512$ cubic inches.

4. **11.** The distance between two lines is equivalent to the distance from one line to any point on the other line. We compute the distance from (0,-5) to the line $x - 2y + 20 = 0$ using $d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} = \frac{|1(0) - 2(-5) + 20|}{\sqrt{1^2 + 2^2}} = \frac{|30|}{\sqrt{5}} = 6\sqrt{5}$.

Method 2: An equation of the line containing the point $(0, -5)$ and perpendicular to the given lines is $y = -2x - 5$. This line intersects $y = \frac{1}{2}x + 10$ at $(-6{,}7)$. The distance between $(0, -5)$ and$(-6{,}7)$ is $\sqrt{180} = 6\sqrt{5}$. The required sum is 11.

5. **5.** From the second equation, $f\big(g(20)\big) = 4$. From the first equation,

$f\big(g(20)\big) = \big(g(20)\big)^3 - 121 = 4$. Then $g(20) = 5$.

6. **50.** Use the fact that if k is odd, $\cos(k\pi) = -1$ and if k is even, $\cos(k\pi) = 1$. The given summation $\sum_{k=1}^{50}(-1)^k \cos(k\pi) = -\cos\pi + \cos(2\pi) - \cos(3\pi) + - \cdots + \cos(50\pi) = 50$.