Nassau County Interscholastic Mathematics League

Team Contest   Answers must be integers from 0 to 999 inclusive.   2012 – 2013

Calculators are allowed.

1. Find the smallest positive integer that is not a divisor of 31!

2. Find the sum of the integer solutions of $1 < (x-5)^2 < 100$.

3. Compute the minimum possible sum of digits for a positive integer that is a multiple of 17.

4. One root of $x^2 + kx - 4 = 0$ is the square of the other root. Find the sum of the cubes of the roots.

5. A tetromino is a figure that consists of four unit squares, each of which shares at least one side with at least one of the other three squares. Compute the number of non-congruent tetrominos.

6. If $\sin x - \cos x = \frac{1}{2}$, and $\sin 2x = \frac{m}{n}$, where m and n are relatively prime positive integers, compute $m + n$.

7. A 2 by 3 rectangle is to be covered by 1 by 2 rectangles and 1 by 1 squares. The 1 by 2 rectangles and 1 by 1 squares must not overlap, and their sides must be parallel to the sides of the 2 by 3 rectangle. Compute the number of possible patterns for such coverings.

8. If $\frac{1}{\sqrt[3]{2}-1} = a + \sqrt[3]{b} + \sqrt[3]{c}$, and a, b, and c are positive integers, **with $a < b < c$**, find $100a + 10b + c$.

9. The centers of circles O and P are inside equilateral triangle ABC, and their radii are 1 and 2, respectively. Circles O and P are externally tangent, circle O is tangent to $\overline{AB}$ and $\overline{AC}$, and circle P is tangent to $\overline{AB}$ and $\overline{BC}$. Given that $AB = \sqrt{m} + \sqrt{n}$, where m and n are positive integers, find $m + n$.

10. Find the value of x for which: $\left(\frac{1+\sqrt{5}}{2}\right)^{2012} + \left(\frac{1+\sqrt{5}}{2}\right)^{2013} = \left(\frac{1+\sqrt{5}}{2}\right)^{x}$.

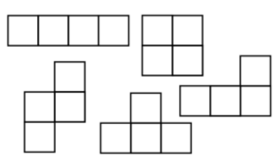
Solutions for Team Contest

1. The positive integers from 1 to 31 inclusive all divide 31!. So do $32, 33, 34, 35$, and 36 because their prime factors are less than 31. But 37 does not divide 31! because 37 is not a prime factor of any of the integers from 1 to 31.

2. Use a transformation. Each of the solutions of $1 < (x-5)^2 < 100$ is 5 greater than a corresponding solution of $1 < x^2 < 100$. There are 16 solutions of the latter inequality, and their sum is 0. So, the sum of the solutions of the original inequality is $16 \cdot 5 = 80$.
   Alternate solution: Since $\sqrt{a^2} = |a|$ for all values of a, the original inequality is equivalent to $1 < |x-5| < 10$. Thus, the distance on the number line between the point whose coordinate is 5 and the point whose coordinate is represented by x is greater than one and less than ten. The integers that meet these requirements satisfy ($x < 4$ or $x > 6$) and $-5 < x < 15$. The sum of these sixteen integers is 80.

3. A number whose digit-sum is 1 must consist of a 1 followed by any number of 0's (including none), but no number ending in 0 whose digit-sum is 1 can be a multiple of 17. If a number whose digit-sum is 2 were to be a multiple of 17, it would have to consist of two 1's with some number of 0's between them. Check on a calculator to find that 100000001 is a multiple of 17. Thus, the minimum digit-sum is 2.
   Alternate solution: Note that $10^2 = 100 \equiv 15 \pmod{17} \equiv -2 \pmod{17}$.
   So, $10^8 = \left(10^2\right)^4 \equiv (-2)^4 \pmod{17} \equiv 16 \pmod{17}$. Therefore,
   $10^8 + 1$ is a multiple of 17.

4. Denote the roots by r and r^2. Then $-4 = r \cdot r^2$, so $r = \sqrt[3]{-4}$, and so $r^2 = \sqrt[3]{16}$. Thus, the requested sum is $-4 + 16 = 12$.

5. There are five. 

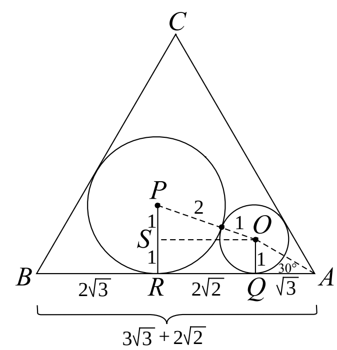
6. Square both sides to obtain $\sin^2 x + \cos^2 x - 2\sin x \cos x = ¼$. This is equivalent to $1 - \sin 2x = ¼$, so $\sin 2x = ¾$ and $m + n = 7$.

7. The number of 1 by 2 rectangles can be 0, 1, 2, or 3. Once all of them are placed, there is only one way to place the 1 by 1 squares. When there are 0 such rectangles, there is 1 possible pattern. There are 7 possible placements for 1 such rectangle because there are 2 placements in each of the two rows and 1 in each if the 3 columns. To place 2 rectangles, consider 3 cases: both horizontal (HH),

both vertical (VV), and one horizontal and one vertical (HV). There are 4 placements of the first type, 3 of the second, and 4 of the third for a total of 11. For 3 rectangles, the cases are HHH, HHV, HVV, and VVV, and there are 0, 2, 0, and 1 placements, respectively, for these cases for a total of 3. Thu,s there are 1 + 7 + 11 + 3 = 22 possible patterns.

8. Let $x = \sqrt[3]{2}$.
Then $\frac{1}{\sqrt[3]{2}-1} = \frac{1}{x-1} = \frac{x^2+x+1}{(x-1)(x^2+x+1)} = \frac{x^2+x+1}{x^3-1} = x^2 + x + 1 = \sqrt[3]{4} + \sqrt[3]{2} + 1$.
Since $a < b < c, a = 1, b = 2,$ and $c = 4$. Thus, $100a + 10b + c = 124$.

9. Let Q and R be the projections of O and P, respectively, onto $\overline{AB}$. Notice that ΔOAQ is a $30° - 60° - 90°$ triangle, so $AQ = \sqrt{3}$. Similarly, $BR = 2\sqrt{3}$. In quadrilateral $OPRQ$, $OP = 1 + 2 = 3$, $PR = 2$, and $OQ = 1$. Let S be the projection of point O onto $\overline{PR}$. Then $PS = 2 – 1 = 1$.
So, $R = OS = \sqrt{3^2 - 1} = 2\sqrt{2}$.
Thus, $AB = 3\sqrt{3} + 2\sqrt{2} = \sqrt{27} + \sqrt{8}$. So, $m + n = 35$.



10. Let $p = \frac{1+\sqrt{5}}{2}$. Then, $p^{2012}(1 + p) = p^x$ or $1 + p = p^{x-2012}$.
So $\frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^{x-2012}$. Therefore, $\frac{3+\sqrt{5}}{2}$ is a power of $\frac{1+\sqrt{5}}{2}$.
So, $x - 2012 = 2$ and $x = 2014$.
<u>Alternate solution</u>: Let $p = \frac{1+\sqrt{5}}{2}$, and let $q = \frac{1-\sqrt{5}}{2}$.
Then $p + q = 1$ and $pq = -1$. So p and q are roots of $x^2 - x - 1 = 0$.

Thus, p must satisfy $p^2 = p + 1$. Multiply both sides by p^{2012} to find

that $p^{2014} = p^{2013} + p^{2012}$, and that therefore $x = 2014$ is a solution of the given equation. The given equation must have a unique solution because $f(x) = p^x$ is an increasing function whose range is the set of positive real numbers.