Nassau County Interscholastic Mathematics League

Contest #2 Answers must be integers from 0 to 999 inclusive. 2012 – 2013

Calculators are allowed.

**Time: 10 minutes**

1. In quadrilateral $ABCD$, $\overline{AB} \perp \overline{BC}$, diagonal $\overline{AC} \perp \overline{CD}$, $AB = 9, BC = 3\sqrt{7}$, and $CD = 5$. Compute $AD$.

2. A student wishes to divide a group of eight teachers into two groups of four. How many such divisions are possible?

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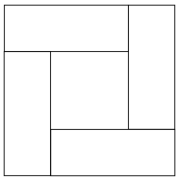
**Time: 10 minutes**

3. The diagram (which is not drawn to scale) shows four congruent rectangles arranged so that they determine two squares. The areas of the squares are 25 and 121. Find the area of one of the rectangles.



4. Let $r$ be the positive root of $x^2 = x + 1$. Given that $r^3 = m + \sqrt{n}$, where $m$ and $n$ are positive integers, compute $m + n$.

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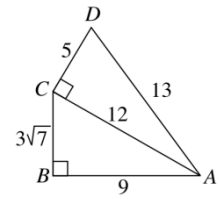
**Time: 10 minutes**

5. Compute the number of positive integer divisors of $10^9$ that do not end in 0.

6. In quadrilateral $ABCD, AB = 6, BC = 7, CD = 8\sqrt{3}, m\measuredangle B = 90°$ and $m\measuredangle C = 120°$ Given that $AD^2 = m + n\sqrt{p}$, where $m$, $n$, and $p$ are positive integers and $p$ is not divisible by the square of a prime, find $m + n + p$.

## Solutions for Contest #2




1. Use the Pythagorean Theorem in $\Delta ABC$ to conclude that $AC = 12$, then use it again in $\Delta ACD$ to find that $AD = 13$.

2. There are $\binom{8}{4} = 70$ four-element subsets of the set of eight teachers. Thus, there are $\frac{70}{2} = 35$ of the desired divisions because each subset and its complement correspond to the same division.

3. Let the dimensions of one of the rectangles be $a$ and $b$ with $a > b$. Then $a + b = 11$ and $a - b = 5$. Solve the system to find that $a = 8$ and $b = 3$. The requested area is therefore 24.

4. Use the Quadratic Formula to find that $r = \frac{1+\sqrt{5}}{2}$. The given equation implies that $r^2 = r + 1$. Multiply by $r$ to obtain $r^3 = r^2 + r = r + 1 + r = 2r + 1 = 2 + \sqrt{5}$. Thus, $m + n = 7$.
   <u>Alternate solution</u>: Let $r = \frac{1+\sqrt{5}}{2} \rightarrow r^2 = \frac{3+\sqrt{5}}{2} \rightarrow r^3 = r^2 \cdot r = 2 + \sqrt{5}$. Again, $m + n = 7$.

5. The prime factors of a divisor of $10^9$ must consist of 2's and/or 5's, so the divisors that do not end in 0 must have only factors of 2 or only factors of 5. That is, they must be of the form $2^a$ or $5^a$, where $0 \le a \le 9$. Because $2^0 = 5^0 = 1$, there are $10 + 10 - 1 = 19$ of the desired divisors.

6. Draw the line through $A$ that is parallel to $\overline{BC}$, and let $E$ be the point where the line intersects $\overline{CD}$. Let $F$ be the projection of $C$ onto $\overline{AE}$. Then $ABCF$ is a rectangle and $\Delta CEF$ is $30° - 60° - 90°$, so $AF = 7, CF = 6, FE = 2\sqrt{3}$, and $CE = 4\sqrt{3}$. Hence, $ED = 4\sqrt{3}$. Apply the Law of Cosines in $\Delta AED$ to find that $AD^2 = 133 + 56\sqrt{3}$. Thus, $m + n + p = 192$.